The Vlasov limit and its fluctuations for a system of particles which interact by means of a wave field

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Version of: February 7, 2008.

Abstract

In two recent publications, [Commun. PDE **22**, 307–335 (1997), Commun. Math. Phys. **203**, 1–19 (1999)], A. Komech, M. Kunze and H. Spohn studied the joint dynamics of a classical point particle and a wave type generalization of the Newtonian gravity potential, coupled in a regularized way. In the present paper the many-body dynamics of this model is studied. The Vlasov continuum limit is obtained in form equivalent to a weak law of large numbers. We also establish a central limit theorem for the fluctuations around this limit.

1   Introduction

In recent years, significant progress has been made on the Cauchy problem of relativistic kinetic theory.[1] The special-relativistic Vlasov–Maxwell equations [LuVl50, Vla61], with applications in theories of astrophysical plasma waves [SchJ73, Jan77], are treated with rigor in [Hor86, GlSt86, GlSt87a, GlSt87b, GlSch88, dPLi89b, Rei90, BGP00, BGP03, KlSt03]; the general-relativistic Vlasov–Einstein equations [Ehl71, Ehl73], which play a rôle in models of cosmological evolutions [Ber88], have been treated rigorously in [ReRe92, Ren94, RRSch95, Rei95]; see also [AnTo99, Ang00]. The relativistic Vlasov–Maxwell and Vlasov–Einstein equations both reduce in their strictly non-relativistic limits to corresponding Vlasov–Poisson equations [Vla38], for which the classical Cauchy problem has been settled [Pfa89, Sch91, Pfa92]. Much of the special-relativistic material is reviewed in [Gla96], the non-relativistic results in [Rei97].

Progress on the microscopic foundations of all these Vlasov models has been lagging behind in comparison. Regularized Vlasov–Poisson equations have been derived through a continuum limit for a family of classical N-body problems with regularized Coulomb and Newton interactions, see [NeWi74, Neu85] and [BrHe77]. In [BrHe77] also a law of large numbers (LLN) and a central limit theorem (CLT) for the fluctuations around this Vlasov limit are proven; see also [Spo91, CIP91] for further discussions. The global regularity results of [Pfa89] should definitely allow one to remove the regularization after the Vlasov limit of the regularized N-body dynamics has been taken and to obtain the proper (i.e. non-regularized) Vlasov–Poisson equations [Vla38], but we are not aware of work where this has been done explicitly. In any event, while mathematically clean, physically such a derivation of the proper Vlasov–Poisson equations is still far from satisfactory, for it uses the wrong order of limits, physically speaking. The regularization should actually be removed while taking the Vlasov limit for the regularized dynamical system, which likewise seems feasible with current techniques, but as far as we know has not yet been done either; however, see [KuRe01a, KuRe01b] for relevant work on the expected radiation-reaction corrections to Vlasov–Poisson and other Vlasov models. Another open question is whether one can obtain the proper Vlasov–Poisson equations directly from the classical Newtonian N-body problem for Newton or Coulomb interactions without any regularization, essentially because the classical N-body problem is still not well-controlled. For a derivation of the classical Vlasov–Poisson equations from a regularized quantum mechanical N-particles model,

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[1]Beside these physical Vlasov models, also the "relativistic Vlasov–Poisson equations" [GlSch85] and more recently the Vlasov–Nordström equations [CaRe03, CaRe04] have been studied.

see [NaSe81]; we also mention a derivation of a Schrödinger–Poisson model from an N-particles quantum model without regularization, see [BEGMY02]. While there is thus plenty of mathematical work left to be done on the microscopic foundations of the non-relativistic Vlasov–Poisson equations, their status is moderately well established. The microscopic foundations of the relativistic Vlasov–Maxwell and Vlasov–Einstein equations by contrast seem not to have been established with rigor in any form.

To bring about a modest change in the state of affairs of the microscopic foundations of relativistic Vlasov theory, in this paper we prove a LLN and a CLT for a regularization of the following (almost) special-relativistic generalization of the Vlasov–Poisson equations for a self-gravitating system,[2] comprising the continuity equation

$$\left(\partial_t + v \cdot \partial_x - \partial_x \phi(x,t) \cdot \partial_p\right) f(x,p,t) = 0 \tag{1}$$

on x, p phase space, where

$$v = \frac{p}{\sqrt{1 + |p|^2}} \tag{2}$$

is the velocity of a generic particle with momentum p and empirical mass of unity, and the inhomogeneous wave equation

$$\Box \phi(x,t) = \rho(x,t) \tag{3}$$

on x space, where $\Box = -\partial_t^2 + \partial_x^2$ is the d'Alembertian,[3] and where

$$\rho(x,t) = \int_{\mathbb{R}^3} f(x,p,t) \mathrm{d}p \tag{4}$$

is the normalized density of particles attributed to the space point $x \in \mathbb{R}^3$ at time $t \in \mathbb{R}$. Clearly, $\phi(x,t)$ is a wave-like generalization of the Newtonian gravity potential generated by $\rho(x,t)$, and $f(x,p,t)$ in turn is the normalized density of particles attributed to the phase-space point $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$ at time $t \in \mathbb{R}$. We remark that although a normalized density $f(\,.\,,\,.\,,t)$ formally satisfies the definition of a probability

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[2]We use natural dimensionless units to avoid burdening the equations with irrelevant dimensional constants. Conversion of equations (1)–(4) to the more conventional Gaussian units for a "gravitational" system is effected by replacing $t \mapsto Nct$, $x \mapsto Nx$, $v \mapsto v/c$, $p \mapsto p/(mc)$, $\phi \mapsto \phi/c^2$, $\rho \mapsto 4\pi Gm\rho/(Nc^2)$, $f \mapsto 4\pi Gcm^4 f/N$; here, c is the speed of light, G is Newton's constant of universal gravitation, N is the total number of particles in the system, and m is the empirical mass of a particle. Note that ρ and f retain their normalization as probability densities on $\mathbb{R}^3$ and $\mathbb{R}^6$.

[3]We write $\partial_x^2 \equiv \mathrm{div\,grad}$ rather than Δ, for Δ is used with a different meaning later on.

density function, $f(\,.\,,\,.\,,t)$ is more properly thought of as (an approximation to) the actual empirical phase space density of particles for an individual system.

It is to be noted that our set of equations (1), (2), (3), (4) is not meant to be taken physically seriously in itself; in particular, the model is not manifestly Lorentz-covariant (more on that in a moment). Its derivation from a microscopic model mainly serves as a simpler primer for the derivation of the special-relativistic Vlasov–Maxwell equations, which we undertake in a sequel to this paper. Indeed, the model (1), (2), (3), (4) is a simple truncation of the usual set of special-relativistic Vlasov–Maxwell equations for a single species of (say, positive unit charge) particles, obtained as follows:[4] in the Vlasov–Maxwell equations, the electromagnetic fields E and B are expressed in terms of the electromagnetic potentials ϕ and A as $B = \nabla \times A$ and $E = -\partial_t A - \nabla \phi$, gauged by the Lorentz–Lorenz condition $\partial_t \phi + \nabla \cdot A = 0$; one then purges the inhomogeneous vector wave equation for A and all terms involving A (or rather its derivatives) in the Lorentz force. Curiously, and contrary to what one might have naively thought, this mutilation of the Vlasov–Maxwell equations does not result in a model which approximates quasi-electrodynamical behavior without magnetic fields, but in one which rather mimics some quasi-gravitational system, for in the strictly non-relativistic limit the model formally reduces to the Vlasov–Poisson equations for a Newtonian gravitational system.

We remark that the replacement $\partial_x \phi(x,t) \to \partial_x \phi(x,t)/\sqrt{1+|p|^2}$ in (1) results in an essentially Lorentz-covariant model with scalar interaction field ϕ. We say 'essentially' because this modification of equations (1)–(4) is still not manifestly Lorentz-covariant when ϕ is interpreted as a Lorentz scalar field, for the right-hand side of (3) when taken face value is the time component of a Minkowski vector. However, the model becomes manifestly Lorentz-covariant when this set of equations is supplemented by the constraint $V \equiv \int_{\mathbb{R}^3} j(x,t)\mathrm{d}x = 0$, where

$$j(x,t) = \int_{\mathbb{R}^3} v f(x,p,t)\mathrm{d}p \tag{5}$$

is the mass current vector density, and the right-hand side of (3) is interpreted as the Minkowski scalar $M\rho - Vj$ in the center-of-mass frame, in which $V = 0$ and $M \equiv \int_{\mathbb{R}^3} \rho(x,t)\mathrm{d}x = 1$. As does our truncated Vlasov–Maxwell model, the Vlasov model with a factor $1/\sqrt{1+|p|^2}$ multiplying $\partial_x \phi(x,t)$ in (1) formally reduces to the

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[4]To obtain this "truncated Vlasov–Maxwell system" in the conventional Gaussian units, replace $t \mapsto Nct$, $x \mapsto Nx$, $v \mapsto v/c$, $p \mapsto p/(mc)$, $\phi \mapsto -e\phi/(mc^2)$, $\rho \mapsto 4\pi e^2 \rho/(Nmc^2)$, $f \mapsto 4\pi e^2 cm^2 f/N$ in our dimensionless equations (1), (2), (3), (4); here, c, m, and N have the same meaning as for the "gravitational" system, while e is the empirical unit of charge of a particle.

Vlasov–Poisson equations for a Newtonian gravitational system in the strictly non-relativistic limit. While this model has a number of interesting features, we will not pursue it further here because it is less relevant to the Vlasov–Maxwell equations.

Ideally, we would like to prove that the kinetic equations (1)–(4) constitute a LLN for the dynamics of an atomistic system of N classical point particles that interact by means of a wave gravity field. The natural candidate for this atomistic system is suggested by "atomizing" the characteristic system for (1), which reads

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{p}{\sqrt{1 + |p|^2}} \, , \tag{6}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\nabla\phi\left(q, t\right) \, , \tag{7}$$

with $\phi(x, t)$ the wave field for (1)–(4). Thus, interpreting f as an empirical atomic measure of N classical point particles, having positions $q_i^{(N)}(t)$ and momenta $p_i^{(N)}(t)$ at time t, these particle motions satisfy the characteristic equations of motion, viz.

$$\dot{q}_i^{(N)}(t) = \frac{p_i^{(N)}(t)}{\sqrt{1 + |p_i^{(N)}(t)|^2}} \, , \tag{8}$$

$$\dot{p}_i^{(N)}(t) = -\nabla\phi^{(N)}\left(q_i^{(N)}(t), t\right) \, , \tag{9}$$

for a $\phi^{(N)}$ which satisfies the inhomogeneous wave equation

$$\Box\phi^{(N)}(x, t) = \tfrac{1}{N}\sum_{i=1}^{N}\delta_{q_i^{(N)}(t)}(x) \, . \tag{10}$$

Unfortunately, this system of equations has only a symbolic character, at best. Since $\phi^{(N)}$ is a distributional solution of (10) which is not in $H^1(B)$ for any open ball B containing the location of a point particle, each particle is surrounded by an infinite field energy which equips the particles with an infinite inertia (via Einstein's $E = mc^2$); furthermore, the right-hand side of (9) is not well-defined. Infinite self-interaction terms are encountered also if one applies the above line of reasoning in the context of the microscopic foundations of the strictly non-relativistic Vlasov–Poisson equations, but in that case the self-interactions are not dynamical, and simply discarding them formally yields a locally well-defined and consistent dynamical system. In a local relativistic theory such a formal omission of the self-interaction terms is not justified because of the dynamical radiation-reaction. Hence, before any classical microscopic derivation based on point particles can be attempted for the relativistic Vlasov–Maxwell, Vlasov–Einstein, and for that matter also for the simpler Vlasov equations considered here, one

first has to overcome the even harder conceptual problem of setting up a well-defined microscopic relativistic model with point particles. While this is being sorted out,[5] it is still of interest to study the obvious amelioration of the infinite self-interactions dilemma by regularizing the ill-defined point particle models [Spo04].

In this vein, we follow [KSpK97, KKSp99], who discussed a regularized version of the symbolic equations (8), (9), (10) with $N = 1$. They "smear" the instantaneous location $q(t) \in \mathbb{R}^3$ of a particle with a probability density function $\varrho(\,.\,) : \mathbb{R}^3 \to \mathbb{R}_+$. Consistency requires that in Newton's equation, the gradient of ϕ for point sources is replaced by the ϱ-average of the gradient of a ϕ_ϱ for ϱ-averaged point sources. The global existence and uniqueness of the dynamics for the regularized microscopic model with $N = 1$ [KSpK97, KKSp99] is readily extended to arbitrary N, with uniform control in t. It should be noted that the regularization just described is non-relativistic.

Interestingly, one of the caveats of the similarly regularized electromagnetic models discussed in [KoSp00, KuSp00a, KuSp00b, KuSp00c, BaDu01] that was pointed out in [Kie99] does not occur in the regularized scalar model of [KSpK97, KKSp99]. Namely, in contrast to what is the case in the electromagnetic models, the a-priori density function ϱ does *not* act as a "source or sink" for the conventional scalar-field angular momentum. Thus, conservation of angular momentum holds in its conventional form and does not need to be rescued through the cosmetical surgery of associating to each particle a spin variable (cf. Appendix A.3 of [ApKi01] for the electromagnetic models).

Our main objective in this paper then is to show that the corresponding regularization of the Vlasov model (1)–(4) governs a LLN for the regularization of the microscopic N particles model with wave gravity interactions (8), (9), (10). To achieve this goal we adapt the strategies of [NeWi74, Neu85], and [BrHe77] from the Vlasov–Poisson to our system of equations; see [Spo91] for an account of Neunzert's proof, and [FiEl98] for an application to a wave modes truncation of the Vlasov equations of plasma physics. The limit $N \to \infty$ not only yields a LLN for the regularized Vlasov equations, but also their well-posedness globally in time. By adapting the strategy of [BrHe77] we also establish a CLT for the fluctuations around the Vlasov limit. It goes without emphasis that these "adaptations" involve plenty of technical and conceptional innovations.

The removal of the regularization has to be addressed at a later time. We expect violations of Lorentz symmetry caused by the finite support of ϱ to vanish when the regularization is removed, either after the Vlasov limit has been taken or along with it. Should this expectation turn out to be unfounded, it would become pointless to try to derive the relativistic Vlasov–Maxwell equations along the lines developed here.

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[5]For recent progress on relativistic microscopic classical electromagnetic theory, see [Kie04].

2 The regularized field & N-body problem

Let $C_c^\infty(\mathbb{R}^3)$ denote the infinitely many times continuously differentiable functions with compact support. In the following, it is assumed that $\varrho(\,.\,) \in C_c^\infty(\mathbb{R}^3)$. For convenience we will also demand that ϱ is radially symmetric and decreasing. For technical reasons [KSpK97] a Wiener condition (positive Fourier transform) needs to be imposed on ϱ.

We introduce the abbreviation $\int = \int_{\mathbb{R}^3}$ and the convolution notations

$$(\varrho * g)(x) = \int \varrho(y-x)g(y)\mathrm{d}y\,, \tag{11}$$

$$(\varrho * \nabla g)(x) = \int \varrho(y-x)\partial_y g(y)\mathrm{d}y\,, \tag{12}$$

$$(\varrho\mathrm{Id} * \times \nabla g)(x) = \int \varrho(y-x)(y-x)\times\partial_y g(y)\mathrm{d}y\,, \tag{13}$$

where $g : \mathbb{R}^3 \to \mathbb{R}$ is any scalar function the derivative of which is in $L^2(\mathbb{R}^3)$.

2.1 The dynamical system

We begin by listing the first-order evolution equations which define the regularized microscopic dynamical model. Incidentally, the model can be viewed as a Hamiltonian system, on which we briefly comment at the end of the next subsection.

Regularizing the inhomogeneous wave equation for the microscopic wave gravity potential with point particle sources gives an inhomogeneous wave equation for the regularized wave gravity potential. Recast as a first-order system for the canonically conjugate scalar field variables[6] $\psi^{(N)}(\,.\,,t) \in \dot{H}^1(\mathbb{R}^3)$ and $\varpi^{(N)}(\,.\,,t) \in L^2(\mathbb{R}^3)$ satisfying

$$\psi^{(N)}(x,0) = -1/4\pi|x| \tag{14}$$

$$\varpi^{(N)}(x,0) = 0 \tag{15}$$

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[6]We recall that the homogeneous Sobolev spaces $\dot{H}^k(\mathbb{R}^d)$ are defined as the closure of $C_c^k(\mathbb{R}^d)$ w.r.t. $\|u\|_{\dot{H}^k}^2 = \sum_{|\alpha|=k} \|D^\alpha u\|_{L^2}^2$, where $C_c^k(\mathbb{R}^d)$ in turn denotes the k times classically differentiable functions with compact support, and α is a multi-index [GiTr01]. The reason for why we do not work with $H^1(\mathbb{R}^3)$ is (14): functions in $\dot{H}^1(\mathbb{R}^3)$ satisfying (14) are not in $L^2(\mathbb{R}^3)$. However, alternatively we could work with the affine Sobolev space $\{\psi : \psi + \frac{1}{4\pi}\varrho * |\,.\,|^{-1} \in L^2(\mathbb{R}^3) \ \& \ \nabla\psi \in L^2(\mathbb{R}^3)\}$ with seminorm $\|\psi\|^2 = \|\nabla\psi\|_{L^2}^2 + \|\psi + \frac{1}{4\pi}\varrho * |\,.\,|^{-1}\|_{L^2}^2$.

outside a closed ball $B_R \subset \mathbb{R}^3$ which contains the initial locations of the N particles and the supports of their regularizations, the inhomogeneous wave equation becomes

$$\partial_t \psi^{(N)}(x,t) = \varpi^{(N)}(x,t) \,, \tag{16}$$

$$\partial_t \varpi^{(N)}(x,t) = \partial_x^2 \psi^{(N)}(x,t) - (\varrho * \rho_t^{(N)})(x) \,, \tag{17}$$

with

$$\rho_t^{(N)}(\,.\,) = \tfrac{1}{N} \sum_{i=1}^{N} \delta_{q_i^{(N)}(t)}(\,.\,) \,; \tag{18}$$

derivatives are understood in the sense of distributions. Note that for given trajectories $t \mapsto q_i^{(N)}(t), i = 1, ..., N$, we have just $\psi^{(N)}(x,t) = (\varrho * \phi^{(N)}(\,.\,,t))(x)$ with $\phi^{(N)}$ solving (10). For $i = 1, ..., N$, the evolution equations for the i-th particle's canonically conjugate positions $q_i^{(N)}(t) \in \mathbb{R}^3$ and momenta $p_i^{(N)}(t) \in \mathbb{R}^3$ at time t, are Einstein's law relating relativistic momentum to velocity,

$$\dot{q}_i^{(N)}(t) = \frac{p_i^{(N)}(t)}{\sqrt{1 + |p_i^{(N)}(t)|^2}} \,, \tag{19}$$

and Newton's law of motion,

$$\dot{p}_i^{(N)}(t) = - \left(\varrho * \nabla\psi^{(N)}(\,.\,,t)\right) \left(q_i^{(N)}(t)\right) \,. \tag{20}$$

A complete specification at time $t \in \mathbb{R}$ of all the first-order evolutionary variables $\left(q_1^{(N)}(t), p_1^{(N)}(t); ...; q_N^{(N)}(t), p_N^{(N)}(t); \psi^{(N)}(\,.\,,t), \varpi^{(N)}(\,.\,,t)\right)$ constitutes a *physical state* in this model. To shorten the notation, we frequently write $z_k^{(N)}(t)$ for the particle variables $(q_k^{(N)}(t), p_k^{(N)}(t))$ and $\mathbf{z}_t^{(N)}$ for $(z_1^{(N)}(t), ..., z_N^{(N)}(t))$; furthermore $\zeta_t^{(N)}$ for the wave field variables $(\psi^{(N)}(\,.\,,t), \varpi^{(N)}(\,.\,,t))$, yet sometimes $\zeta[\mathbf{z}_0^{(N)}]$ rather than $\zeta_0^{(N)}$ for the initial fields when we want to emphasize their dependence on the initial data $\mathbf{z}_0^{(N)}$ rather than merely on N; finally, we frequently write $\mathfrak{z}_t^{(N)}$ for the physical state at time t, viz.

$$\mathfrak{z}_t^{(N)} := \left(\mathbf{z}_t^{(N)}, \zeta_t^{(N)}\right) \,. \tag{21}$$

The space of all possible physical states is known as the *system phase space*. To conveniently adapt some results of [KSpK97], $\Gamma^{(N)}$ is given Hilbert space topology by taking the Hilbert space direct sum of the particle and the field Hilbert spaces,

$$\Gamma^{(N)} = \underbrace{\mathbb{R}^3 \oplus \ldots \oplus \mathbb{R}^3}_{2N \text{ terms}} \oplus \dot{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \,, \tag{22}$$

equipped with the conventional Hilbert space inner product $\langle\,.\,,.\,\rangle$ implied by (22). The subset of $\Gamma^{(N)}$ on which (14), (15) is satisfied is denoted $\Gamma_{\mathrm{B}}^{(N)}$.

The Hilbert space topology of $\Gamma^{(N)}$ is of course equivalent to the Banach space topology for (22) interpreted as a Banach space direct sum, but the Hilbert space topology is indeed more natural for the N-body plus field dynamics. In contrast, a Banach space topology is the natural one for the Vlasov model which we discuss in section 3.

We remark that, while $\dot{H}^1(\mathbb{R}^3)$ and $L^2(\mathbb{R}^3)$ allow quite rough fields $\psi^{(N)}(\,.\,,t)$ and $\varpi^{(N)}(\,.\,,t)$, any roughness would be inherited from the initial data. To have strong solutions of the wave equation in our case, we demand $\psi(\,.\,,0) \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3)$, rather than the usual $\psi(\,.\,,0) \in H^2(\mathbb{R}^3)$; cf. [Ika00]. Higher regularity, e.g. as for classical solutions, can also be obtained by the usual bootstrapping, if desired.

2.2 The conservation laws

The conventional conservation laws for mass, momentum, angular momentum, and energy are satisfied for sufficiently regular solutions of the dynamical system. To state the conservation laws, we introduce several functionals on the system phase space of generic states $(z_1, ..., z_N, \zeta) = (\mathbf{z}^{(N)}, \zeta) =: \mathfrak{z}^{(N)} \in \Gamma^{(N)}$.

The mass functional, for $\rho^{(N)}$ given in (18) with generic q_i, is given by

$$\mathcal{M}\left(\mathfrak{z}^{(N)}\right) = \int \varrho * \rho^{(N)} \mathrm{d}x\,, \tag{23}$$

the momentum functional by

$$\mathcal{P}\left(\mathfrak{z}^{(N)}\right) = \frac{1}{N}\sum_{i=1}^{N}p_i - \int \varpi \partial_x \psi \mathrm{d}x\,, \tag{24}$$

the angular momentum functional by

$$\mathcal{J}\left(\mathfrak{z}^{(N)}\right) = \frac{1}{N}\sum_{i=1}^{N}q_i \times p_i - \int (x \times \partial_x \psi)\varpi \mathrm{d}x\,, \tag{25}$$

and the energy functional by

$$\mathcal{H}\left(\mathfrak{z}^{(N)}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\sqrt{1 + |p_i|^2} + (\varrho * \psi)(q_i)\right) + \frac{1}{2}\int \left(|\partial_x \psi|^2 + |\varpi|^2\right) \mathrm{d}x\,. \tag{26}$$

We note that $\mathcal{M}, \mathcal{P}, \mathcal{H}$ are well-defined on all of $\Gamma^{(N)}$, while $\mathcal{J}$ is well-defined only on a subset of $\Gamma^{(N)}$; in particular, $\mathcal{J}$ is well-defined on $\Gamma_{\mathrm{B}}^{(N)}$.

REMARK 2.1 *The energy functional (26) furnishes the Hamiltonian for the regularized dynamical system. It is readily verified that the Hamiltonian system, $\frac{d}{dt} q_i^{(N)} = \partial \mathcal{H} / \partial p_i^{(N)}$ and $\frac{d}{dt} p_i^{(N)} = -\partial \mathcal{H} / \partial q_i^{(N)}$, together with $\partial_t \psi^{(N)} = \delta \mathcal{H} / \delta \varpi^{(N)}$ and $\partial_t \varpi^{(N)} = -\delta \mathcal{H} / \delta \psi^{(N)}$, coincides with the evolution equations for the wave gravity potential and the particles.*

A map $t \mapsto \mathfrak{z}_t^{(N)} \in C^1(\mathbb{R}, \Gamma^{(N)})$ satisfying our microscopic scalar wave gravity equations will be called a $\Gamma^{(N)}$-strong solution.

PROPOSITION 2.2 *For any sufficiently regular (in particular, a $\Gamma^{(N)}$-strong) solution $t \mapsto \mathfrak{z}_t^{(N)}$ of the microscopic scalar wave gravity system, we have*

$$\mathcal{M} \left(\mathfrak{z}_t^{(N)} \right) = M \,, \tag{27}$$
$$\mathcal{P} \left(\mathfrak{z}_t^{(N)} \right) = P \,, \tag{28}$$
$$\mathcal{H} \left(\mathfrak{z}_t^{(N)} \right) = E \,, \tag{29}$$

with M, P, E independent of time; in particular, $M = 1$. If $\mathfrak{z}_0^{(N)} \in \Gamma_{\mathrm{B}}^{(N)}$, then also

$$\mathcal{J} \left(\mathfrak{z}_t^{(N)} \right) = J \,, \tag{30}$$

with J independent of time.

Proof of Proposition 2.2. Proposition 2.2 is proved in the appendix as a special case of the conservation laws in Theorems 3.2 and 3.3 of subsubsection 3.2.1.   Q.E.D.

REMARK 2.3 *One may contemplate attaching also an Euler spin variable $s_i^{(N)}(t) \in \mathbb{R}^3$ at time t to the i-th particle, the (non-relativistic) evolution equations for $s_i^{(N)}(t)$ being just Euler's equations for a degenerate gyroscope, viz.*

$$\dot{s}_i^{(N)}(t) = - \left(\varrho \mathrm{Id} \ast \times \nabla \psi^{(N)}(\,.\,,t) \right) \left(q_i^{(N)}(t) \right) \,. \tag{31}$$

However, standard identities of vector analysis and the radial symmetry of ϱ yield for the (negative of the) field torque on the i-th particle

$$\left(\varrho \mathrm{Id} \ast \times \nabla \psi^{(N)}(\,.\,,t) \right)(x) = \int \partial_y \times [\varrho(y-x)(y-x)\psi^{(N)}(y,t)] \mathrm{d}y \equiv 0, \tag{32}$$

the vanishing as a result of one of Green's theorems and the compact support of ϱ. Hence, each $s_i^{(N)}(t)$ is itself a constant of the motion. Moreover, since there is no feedback loop from $s_i^{(N)}(t)$ to the particle-field dynamics, the introduction of spin into this model is uncalled for.

2.3 Global existence and uniqueness

In this subsection we extend the single particle global existence and regularity results of [KSpK97], [KKSp99] to the many-body problem. To get started, one needs decent a-priori bounds on the norms of the various dynamical quantities.

2.3.1 A-priori bounds without invoking conservation laws

We begin with a-priori bounds that can be obtained without invoking the conservation laws. It is trivially clear by the upper bound 1 on their speeds that the positions of the particles are bounded above linearly in t. In the following we recall the familiar linear in t a-priori estimate for the field norms, and a bound on the momenta quadratic in t.

LEMMA 2.4 Let $\left(q_1^{(N)}(\,.\,), ..., q_N^{(N)}(\,.\,)\right) \in C^{0,1}(\mathbb{R}, \mathbb{R}^{3N})$ be a given Lipschitz-continuous curve, its components having Lipschitz constant < 1, and let $\zeta_. \in C^1(\mathbb{R}, (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$ be a strong solution of (16), (17), satisfying conditions (14) (15). Then at any $t \in \mathbb{R}$,

$$\max\{\|\psi^{(N)}(\,.\,,t)\|_{\dot{H}^1}, \|\varpi^{(N)}(\,.\,,t)\|_{L^2}\} \leq (2\mathcal{E}_{\mathrm{W}}(\zeta_0^{(N)}))^{1/2} + \|\varrho\|_{L^2}|t|, \qquad (33)$$

where $\mathcal{E}_{\mathrm{W}}(\zeta_t) = \frac{1}{2}\int(|\partial_x\psi(\,.\,,t)|^2 + |\varpi(\,.\,,t)|^2)\mathrm{d}x$ is the wave field energy at time t.

REMARK 2.5 The a-priori bound (33) extends to the strong solution of the wave equation for any subluminal source $\varrho * \rho \in C^0(\mathbb{R}, C_0^\infty(\mathbb{R}^3))$.

Proof of Lemma 2.4: By hypothesis, $t \mapsto \zeta_t^{(N)} \in C^1(\mathbb{R}, (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$ is a strong solution of the wave equation with source $t \mapsto \varrho * \rho_t^{(N)} \in C^0(\mathbb{R}, C_c^\infty(\mathbb{R}^3))$ moving at speeds less than light; hence, $t \mapsto \mathcal{E}_{\mathrm{W}}(\zeta_t^{(N)})$ is differentiable. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathrm{W}}(\zeta_t^{(N)}) = -\int \varpi^{(N)}(t, x)(\varrho * \rho_t^{(N)})(x)\mathrm{d}x \leq \|\varpi^{(N)}(\,.\,,t)\|_{L^2}\|\varrho * \rho_t^{(N)}\|_{L^2}$$

$$\leq \|\varrho\|_{L^2}\left(2\mathcal{E}_{\mathrm{W}}(\zeta_t^{(N)})\right)^{1/2}, \qquad (34)$$

the first inequality by Cauchy–Schwarz, while for the second one we used the estimate $\|\varrho * \rho_t^{(N)}\|_{L^2}^2 \leq \sup_{x\in\mathbb{R}^3}\{(\varrho * \varrho)(x)\} = \|\varrho\|_{L^2}^2$, as well as the obvious estimate $\|\varpi^{(N)}(\,.\,,t)\|_{L^2} \leq \left(2\mathcal{E}_{\mathrm{W}}(\zeta_t^{(N)})\right)^{1/2}$ implied by the definition of $\mathcal{E}_{\mathrm{W}}$. Dividing (34) by $\left(2\mathcal{E}_{\mathrm{W}}(\zeta_t^{(N)})\right)^{1/2}$ and integrating over t gives

$$(2\mathcal{E}_{\mathrm{W}}(\zeta_t^{(N)}))^{1/2} \leq (2\mathcal{E}_{\mathrm{W}}(\zeta_0^{(N)}))^{1/2} + \|\varrho\|_{L^2}t. \qquad (35)$$

The definition of $\mathcal{E}_{\mathrm{W}}$ given in Lemma 2.4 now shows that (35) implies (33).   Q.E.D.

LEMMA 2.6 *Let* $\left(q_1^{(N)}(\,.\,), ..., q_N^{(N)}(\,.\,)\right) \in C^{0,1}(\mathbb{R}, \mathbb{R}^{3N})$ *be a given Lipschitz-continuous curve, its components having Lipschitz constant* < 1, *and let* $\zeta_. \in C^1(\mathbb{R}, (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$ *be a strong solution of the wave equation with source* $t \mapsto \varrho * \rho_t^{(N)} \in C^0(\mathbb{R}, C_c^\infty(\mathbb{R}^3))$. *Suppose* $t \mapsto \left(p_1^{(N)}(t), ..., p_N^{(N)}(t)\right)$ *is a classical solution of (20). Then the momenta at* $t \in \mathbb{R}$, $p_k^{(N)}(t)$, $k = 1, ..., N$, *are bounded by*

$$\max_{1 \leq k \leq N}\{\|p_k^{(N)}(t)\|\} \leq \max_{1 \leq k \leq N}\{\|p_k^{(N)}(0)\|\} + \|\varrho\|_{L^2}(2\mathcal{E}_{\mathrm{W}}(\zeta_0^{(N)}))^{1/2}|t| + \tfrac{1}{2}\|\varrho\|_{L^2}^2 t^2 . \qquad (36)$$

Proof of Lemma 2.6: Use $p(t) = p(0) + \int_0^t \dot{p}(\tilde{t})\mathrm{d}\tilde{t}$, take absolute values, use the triangle inequality, then invoke (20) and estimate

$$| (\varrho * \nabla \psi^{(N)}(\,.\,, t)) \left(q_i^{(N)}(t)\right) | \leq \|\varrho\|_{L^2}\|\psi^{(N)}(\,.\,, t)\|_{\dot{H}^1} , \qquad (37)$$

then recall Lemma 2.4.   Q.E.D.

REMARK 2.7 *We note that (35) is far from optimal, which is evident from the fact that no details of the time dependence of* $\rho_t^{(N)}$ *enter (35); in any event, squaring gives an upper bound on the wave field energy quadratic in t. Similarly, (36) is far from optimal; in any event, its right-hand side provides a quadratic-in-t upper bound on the kinetic energy, "l.h.s.(47)−1." These bounds together with the bound (37) and the asymptotics (14) now also imply that* $|\frac{1}{N} \sum_{i=1}^N (\varrho * \psi^{(N)}(\,.\,, t)) \left(q_i^{(N)}(t)\right)|$, *and therefore finally also the total energy, are both bounded above by* $a + b|t| + ct^2$.

This does not yet exhaust our list of bounds that obtain without invoking conservation laws. The next such bound is nevertheless given its own subsection, for the special role it plays subsequently.

2.3.2   A lower bound on the Hamiltonian functional
To state our lower bound on the Hamiltonian functional, we first define

$$E_\perp := 1 - \frac{1}{8\pi} \iint \frac{\varrho(x)\varrho(y)}{|x - y|}\mathrm{d}x\mathrm{d}y . \qquad (38)$$

Note that the energy value $E_\perp$ depends only on the regularization but not on N.

PROPOSITION 2.8 *The Hamiltonian functional is bounded below by*

$$\mathcal{H}\left(\mathfrak{z}^{(N)}\right) \geq E_\perp \,, \tag{39}$$

independently of N. The bound is attained when $\mathfrak{z}^{(N)}$ is any translation in space of $\mathfrak{z}_\perp^{(N)}$, the state in which for all $k = 1, ..., N$ we have $q_k = 0$, $p_k = 0$, and furthermore $\varpi \equiv 0$ and $\psi \equiv \psi_\varrho$, with

$$\psi_\varrho(x) = -\frac{1}{4\pi} \left(|\,.\,|^{-1} * \varrho \right)(x). \tag{40}$$

(However, note that only the standard ground state satisfies (14).)

The state $\mathfrak{z}_\perp^{(N)}$ will be called the standard ground state of the regularized dynamical system, and (38) will be called the ground state energy.

Proof of Proposition 2.8. For later purposes, we will prove the bound (39) as an upper limit of a one-parameter family of bounds to $\mathcal{H}\left(\mathfrak{z}^{(N)}\right)$. Thus, let $\kappa \in (0, 1]$. Then

$$\mathcal{H}\left(\mathfrak{z}^{(N)}\right) - \tfrac{1-\kappa}{2}\|\psi\|_{\dot{H}^1}^2 = \tfrac{1}{N}\sum_{i=1}^{N}\left(\sqrt{1+|p_i|^2} + (\varrho * \psi)(q_i)\right) + \tfrac{1}{2}\|\varpi\|_{L^2}^2 + \tfrac{\kappa}{2}\|\psi\|_{\dot{H}^1}^2. \tag{41}$$

Discarding the manifestly positive momentum contributions we obtain

$$\mathcal{H}\left(\mathfrak{z}^{(N)}\right) - (1-\kappa)\tfrac{1}{2}\|\psi\|_{\dot{H}^1}^2 \geq 1 + \kappa\tfrac{1}{2}\|\psi\|_{\dot{H}^1}^2 + \tfrac{1}{N}\sum_{i=1}^{N}(\varrho * \psi)(q_i)\,. \tag{42}$$

Minimizing the right-hand side of (42) with respect to ψ now gives

$$\mathcal{H}\left(\mathfrak{z}^{(N)}\right) - (1-\kappa)\tfrac{1}{2}\|\psi\|_{\dot{H}^1}^2 \geq 1 - \tfrac{1}{N^2}\sum_{i=1}^{N}\sum_{k=1}^{N}\frac{1}{8\pi\kappa}\iint \frac{\varrho(q_i - x)\,\varrho(q_k - y)}{|x - y|}\mathrm{d}x\mathrm{d}y\,. \tag{43}$$

The right-hand side of (43) can be minimized w.r.t. the $\{q_i\}_{i=1}^{N}$ by equi-measurable, radially symmetric rearrangement of $\sum_{n=1}^{N}\varrho(\,.\,-q_n)$ centered at the origin. Since ϱ is itself radially symmetric and decreasing, this is achieved by simply translating all q_n to the same position, in particular to the origin. This gives, for all $\kappa \in (0, 1]$,

$$\mathcal{H}\left(\mathfrak{z}^{(N)}\right) - (1-\kappa)\tfrac{1}{2}\|\psi\|_{\dot{H}^1}^2 \geq 1 - \frac{1}{8\pi\kappa}\iint \frac{\varrho(x)\varrho(y)}{|x - y|}\mathrm{d}x\mathrm{d}y. \tag{44}$$

The bound (39) now obtains by taking $\kappa = 1$ in (44) and recalling (38). Straightforward computation of $\mathcal{H}\left(\mathfrak{z}_\perp^{(N)}\right)$ proves that (39) is attained at $\mathfrak{z}_\perp^{(N)}$ and, by the translation invariance in position space of $\mathcal{H}\left(\mathfrak{z}^{(N)}\right)$, also at any translate of $\mathfrak{z}_\perp^{(N)}$.   Q.E.D.

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2.3.3   Bounds invoking conservation laws

Using energy conservation of sufficiently regular solutions, we next bootstrap from the proof of Proposition 2.8 to uniform bounds in t and N on the four major additive contributions to $\mathcal{H}\left(\mathfrak{z}_t^{(N)}\right)$.

LEMMA 2.9 *Let $t \mapsto \mathfrak{z}_t^{(N)}$ be a sufficiently regular (e.g. $\Gamma^{(N)}$-strong) solution of the dynamical system (16)–(20) conserving energy. Then, uniformly in t and N, we have*

$$\left\|\psi^{(N)}(\,.\,,t)\right\|_{\dot{H}^1}^2 \leq 4 + 4E - 8E_\perp\,, \tag{45}$$

$$\left\|\varpi^{(N)}(\,.\,,t)\right\|_{L^2}^2 \leq 2E - 2E_\perp\,, \tag{46}$$

$$\frac{1}{N}\sum_{i=1}^{N}\sqrt{1 + |p_i^{(N)}(t)|^2} \leq 1 + E - E_\perp\,, \tag{47}$$

$$6E_\perp - 3E - 3 \leq \frac{1}{N}\sum_{i=1}^{N}(\varrho * \psi^{(N)}(\,.\,,t))\left(q_i^{(N)}(t)\right) \leq E - 1 \tag{48}$$

Proof of Lemma 2.9. Since $\mathcal{H}\left(\mathfrak{z}_t^{(N)}\right) = E$ is fixed by the Cauchy data $\mathfrak{z}_0^{(N)}$, a simple rewriting of (44) with $\kappa = 1/2$, using the definition (38), gives us (45). As to (46) and (47), $\mathcal{H}\left(\mathfrak{z}_t^{(N)}\right) = E$ and the definition (26) of $\mathcal{H}\left(\mathfrak{z}^{(N)}\right)$ give us the identity

$$\frac{1}{N}\sum_{i=1}^{N}\sqrt{1 + \left|p_i^{(N)}(t)\right|^2} + \tfrac{1}{2}\left\|\varpi^{(N)}(\,.\,,t)\right\|_{L^2}^2 = $$
$$E - \tfrac{1}{2}\left\|\psi^{(N)}(\,.\,,t)\right\|_{\dot{H}^1}^2 - \frac{1}{N}\sum_{i=1}^{N}\left(\varrho * \psi^{(N)}(\,.\,,t)\right)\left(q_i^{(N)}(t)\right)\,. \tag{49}$$

Recalling the minimization steps that lead from (42) to (44) (here with $\kappa = 1$), and the definition (38), we see that the right-hand side of (49) is bounded above, giving

$$\frac{1}{N}\sum_{i=1}^{N}\sqrt{1 + \left|p_i^{(N)}(t)\right|^2} + \tfrac{1}{2}\left\|\varpi^{(N)}(\,.\,,t)\right\|_{L^2}^2 \leq 1 + E - E_\perp\,. \tag{50}$$

Now (46) follows at once from (50) by estimating $|p_i^{(N)}(t)| \geq 0$; to get (47), we instead use $\left\|\varpi^{(N)}(\,.\,,t)\right\|_{L^2}^2 \geq 0$ in (50). Finally, to obtain (48), rewrite the definition (26) of $\mathcal{H}\left(\mathfrak{z}^{(N)}\right)$ into an identity for $\frac{1}{N}\sum_{i=1}^{N}(\varrho * \psi^{(N)}(\,.\,,t))\left(q_i^{(N)}(t)\right)$, then use $\mathcal{H}\left(\mathfrak{z}_t^{(N)}\right) = E$; now the bounds (45), (46), (47) give the first, the positivity of $\left\|\psi^{(N)}(\,.\,,t)\right\|_{\dot{H}^1}^2$, $\left\|\varpi^{(N)}(\,.\,,t)\right\|_{L^2}^2$, and $|p_i^{(N)}(t)|$ the second inequality in (48).   Q.E.D.

REMARK 2.10 *The bounds (46) and (47) happen to be asymptotically sharp when $E \downarrow E_\perp$, in which case they correctly imply that $\|\varpi^{(N)}(\,.\,,t)\|_{L^2} \downarrow 0$ and $|p_i^{(N)}(t)| \downarrow 0$ for all $i = 1, ..., N$. It is to be doubted though that (46) and (47) are sharp for $E > E_\perp$; in any event, certainly (45) and (48) are not sharp (for instance, (45) misses the correct ground state value by a factor 2). Of course, it is a straightforward matter to improve on (45) and (48) by optimizing w.r.t. κ (N.B.: $\kappa = 1/2$ is the optimizer for $E = 1$), and while this does lead to asymptotically sharp upper bounds as $E \downarrow E_\perp$ (in which case $\kappa \uparrow 1$), for $E > E_\perp$ these bounds are still not sharp, but now more cumbersome than (45) and (48). Fortunately, for our purposes any a-priori bounds uniform in t and N will do; hence, we gain by sticking to the simple ones given in Lemma 2.9.*

As a corollary to (47) the particle momenta are bounded above in magnitude. This has an easy but important corollary for the particle speeds, which we state explicitly.

COROLLARY 2.11 *The particle speeds are bounded away from the speed of light, viz.*

$$\max_{i \in \{1, ..., N\}} |\dot{q}_i^{(N)}(t)| \le \sqrt{1 - (1 + N(E - E_\perp))^{-2}}, \tag{51}$$

uniformly in t. In particular, when $E = E_\perp$, then $|\dot{q}_i^{(N)}(t)| = 0$ for all i and N.

Proof of Corollary 2.11. We rewrite (47) as

$$\frac{1}{N} \sum_{i=1}^{N} \left(\sqrt{1 + |p_i^{(N)}(t)|^2} - 1 \right) \le E - E_\perp. \tag{52}$$

Since $\sqrt{1 + |p|^2} - 1 \ge 0$, the bound (52) now implies that for all i,

$$\sqrt{1 + |p_i^{(N)}(t)|^2} - 1 \le N(E - E_\perp), \tag{53}$$

and solving for $|p_i^{(N)}(t)|$ gives, uniformly in t,

$$\max_{i \in \{1, ..., N\}} |p_i^{(N)}(t)| \le \sqrt{(1 + N(E - E_\perp))^2 - 1}, \tag{54}$$

which now yields (51) by inverting the monotone map $|p| \mapsto |v|$ given in (2).   Q.E.D.

Note that for any $E > E_\perp$, (51) does not imply boundedness away from the speed of light of the $|\dot{q}_i^{(N)}(t)|$ uniformly in N; only $\max_i |\dot{q}_i^{(N)}(t)| \le 1$ holds uniformly in N.

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2.3.4 Global existence and uniqueness of solutions

Lemma 2.9 and Corollary 2.11 imply that any energy-conserving solution is represented by a point moving in a weakly compact subset of $\Gamma^{(N)}$, and such solutions do exist.

THEOREM 2.12 *For every* $\mathfrak{z}_0^{(N)} \in \Gamma_B^{(N)}$ *there exists a unique, global strong solution* $t \mapsto \mathfrak{z}_t^{(N)} \in C^1(\mathbb{R}, \Gamma^{(N)})$ *of the Hamiltonian field & N-body problem (16)–(20), satisfying* $\lim_{t\to 0} \mathfrak{z}_t^{(N)} = \mathfrak{z}_0^{(N)}$, *and conserving mass, energy, momentum and angular momentum as stated in Proposition 2.2. For more regular initial data one can bootstrap to correspondingly higher regularity of* $t \mapsto \mathfrak{z}_t^{(N)}$.

Proof of Theorem 2.12. The proof is a largely straightforward adaption to our many-body problem of the proof for a single particle system in [KSpK97]. We remark that our Wiener condition for ϱ is only needed to adapt their proof. The strategy is to first construct local weak solutions conserving energy, then to use the uniform bounds on the norms of the various dynamical quantities that follow from energy conservation (see Lemma 2.9 and its Corollary) to continue to all times. Strong solutions obtain by restricting $\psi(\,.\,,0) \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3)$. Proofs of the conservation laws, which are stated without proof in [KSpK97], are provided in our appendix, for the convenience of the reader.   Q.E.D.

REMARK 2.13 *For the proof of Theorem 2.12 the a-priori bounds in Lemma 2.9 based on energy conservation suffice. The other conservation laws provide additional bounds that may be useful in different contexts. For instance, momentum conservation and the Cauchy–Schwarz inequality give us the uniform bound in* t,

$$\frac{1}{N}|\sum_{i=1}^{N} p_i^{(N)}(t)| \leq |P| + \|\psi^{(N)}(\,.\,,t)\|_{\dot{H}^1}\|\varpi^{(N)}(\,.\,,t)\|_{L^2}\,, \tag{55}$$

while angular momentum conservation, the Cauchy–Schwarz inequality, and the finite wave speed give us the linear bound in t,

$$\frac{1}{N}|\sum_{i=1}^{N} p_i^{(N)}(t) \times q_i^{(N)}(t)| \leq |J| + (R + |t|)\,\|\psi^{(N)}(\,.\,,t)\|_{\dot{H}^1}\|\varpi^{(N)}(\,.\,,t)\|_{L^2}\,, \tag{56}$$

where R *is the radius of the ball* B_R *containing the initial positions of all particles and the supports of their regularizations, and outside of which (14) and (15) hold.*

REMARK 2.14 *As is the case for the regularized Vlasov–Poisson equations [Spo91], the solutions described by Theorem 2.12 map one-to-one into generalized solutions of the regularized wave gravity Vlasov model in which derivatives of* f *are meant in the sense of distributions.*

3 The regularized Vlasov model

In this section we discuss the regularized wave gravity Vlasov model. First, we present the Vlasov equations formally as a continuum model. Next we recall the concept of generalized (distributional) solutions, for which we introduce two suitable topologies, one based on the vague and one on a strong Banach space topology for distributions. The solutions to the field & N-body model of the previous section furnish particular generalized solutions of our Vlasov model in either of the just mentioned topologies. We then prove global existence and uniqueness in the strong Banach space topology of generalized solutions to our regularized wave gravity Vlasov model.

3.1 The dynamical continuum system

As first order system, the inhomogeneous wave equation for the regularized wave gravity potential $\psi(\,.\,,t) \in \dot{H}^1(\mathbb{R}^3)$ and its conjugate variable $\varpi(\,.\,,t) \in L^2(\mathbb{R}^3)$ now reads

$$\partial_t \psi(x,t) = \varpi(x,t) \tag{57}$$

$$\partial_t \varpi(x,t) = \partial_x^2 \psi(x,t) - (\varrho * \rho(\,.\,,t))(x)\,. \tag{58}$$

The initial data $\psi(\,.\,,0) \equiv \psi_0(\,.\,) \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3)$ and $\varpi(\,.\,,0) \equiv \varpi_0(\,.\,) \in L^2(\mathbb{R}^3)$ satisfy

$$\psi_0(x) = -1/4\pi|x|\,, \tag{59}$$

$$\varpi_0(x) = 0 \tag{60}$$

outside some closed ball $B_R \subset \mathbb{R}^3$. The density $\rho(x,t)$ on the r.h.s. in (58) is given by

$$\rho(x,t) = \int f(x,p,t)\mathrm{d}p\,, \tag{61}$$

where $f(\,.\,,\,.\,,t)$ is the normalized particle density function at time t, satisfying the following (continuity) equation on time-position-momentum space $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$\partial_t f(x,p,t) = -\left(\partial_p \sqrt{1 + |p|^2} \cdot \partial_x - \partial_x \left(\varrho * \psi(\,.\,,t)\right)(x) \cdot \partial_p\right) f(x,p,t)\,, \tag{62}$$

with $x \in \mathbb{R}^3$ being the space and $p \in \mathbb{R}^3$ the momentum variable. Initial data $f(\,.\,,\,.\,,0) \equiv f_0(\,.\,,\,.\,)$ for (62) are restricted by the requirement that $\varrho * \rho(\,.\,,0)$ is supported in B_R. As to the appropriate function space, we re-emphasize that in the form stated above, one should think of Vlasov's $f(\,.\,,\,.\,,t)$ as a continuum approximation

to the empirical x, p phase space density of particles for an actual individual N-body system in the large N regime, when fine details of the particles' behaviors become irrelevant on the "macroscopic" scales so that the empirical atomic measure can be well approximated by a function $f(., ., t) \in L^1_{+,1}(\mathbb{R}^6)$ — the subset of $L^1(\mathbb{R}^6)$ consisting of the Radon–Nikodym derivatives f of Borel probability measures $\mu^f(\mathrm{d}x\mathrm{d}p)$ which are absolutely continuous w.r.t. Lebesgue measure. In fact, such functions $f(., ., t)$, the fields $\psi(., t)$ and their formal time derivatives $\varpi(., t)$, would even be expected to have time and space derivatives in the classical sense for all time whenever their initial data are chosen sufficiently regular.

REMARK 3.1 *It is known, but perhaps not well-known, that for sufficiently regular solutions f (say, classical with rapid decay at infinity), the continuity equation (62) can readily be associated with a Hamiltonian $\mathcal{H}_\mathrm{C}$, given $\psi(., t)$ for all t. To obtain the Hamiltonian $\mathcal{H}_\mathrm{C}$ for f given $\psi(., t)$, multiply r.h.s.(62) with a test function $g(., ., t) \in C^1(\mathbb{R}^6)$ of at most polynomial growth in x, p whose t-dependence is yet to be determined, and integrate over $\mathbb{R}^6$; f and g can now be viewed as conjugate variables, with $\partial_t f = \delta\mathcal{H}_\mathrm{C}(f,g)/\delta g$, $\partial_t g = -\delta\mathcal{H}_\mathrm{C}(f,g)/\delta f$. Interestingly, the equation for g is just (62) with g in place of f, and in this sense (62) already is the Hamiltonian system, given the fields. The inhomogeneous wave equation (57), (58) for the fields $\psi(., t)$ and $\varpi(., t)$ is a Hamiltonian dynamical system, given f. The full set of equations (57), (58), (62) becomes a Hamiltonian system with the help of non-canonical Lie brackets, cf. [Mor80, WeMo81, MMW84].*

Our goal is to validate the continuum approximation to the microscopic atomistic dynamics by means of a continuum limit in x, p space (the "Vlasov limit"), supplemented by a law of large numbers and a central limit theorem. To pave the way for the continuum validation, we next recall the concept of generalized solutions.

3.2   Distributional form of the regularized Vlasov model

In order to think of $f(., ., t)$ as the actual atomic measure of an individual N-body system, one has to interpret the derivatives in the sense of distributions. Thus, for given $\psi(., t) \in \dot{H}^1(\mathbb{R}^3)$ and $\varpi(., t) \in L^2(\mathbb{R}^3)$, we implement the idea of distributional derivatives of f in the usual way by multiplying (all of) (62) with any real test function $g(., .) \in C^1_0(\mathbb{R}^6)$ and integrate over $\mathbb{R}^6$ by parts to transfer the partial derivatives w.r.t. x, p onto the smooth g; also, the partial derivative w.r.t. t is pulled out of the integral. So far, $f(., ., t)$ had to be a sufficiently regular function, but nothing now prevents us from allowing $f \in L^1_{+,1}(\mathbb{R}^6)$, the Radon–Nikodym derivative of an absolutely continuous

measure μ^f. The so integrated and manipulated form of (62) remains well-defined even if we replace $\mu^f(\mathrm{d}x\mathrm{d}p)$ with any Borel probability measure $\mu_t(\mathrm{d}x\mathrm{d}p)$. Indeed, let Dg denote any of the partial derivatives of g. Then $Dg \in C_0^0(\mathbb{R}^6)$, where $C_0^0(\mathbb{R}^6)$ is equipped with the uniform norm (a.k.a. sup-norm) $\|Dg\|_{\mathrm{u}} = \sup_{z\in\mathbb{R}^6} |Dg(z)|$. On the other hand, the Borel probability measures $P(\mathbb{R}^6)$ are a subset of $M(\mathbb{R}^6)$, the Banach space of finite signed Radon measures σ (which on $\mathbb{R}^6$ coincide with the finite regular signed Borel measures σ) equipped with the total variation (TV) norm $\|\sigma\|_{\mathrm{TV}} = (|\sigma_+| + |\sigma_-|)(\mathbb{R}^6)$, and $M(\mathbb{R}^6)$ is isometrically isomorphic to $C_0^0(\mathbb{R}^6)^*$, the dual space for (real) $C_0^0(\mathbb{R}^6)$.

In the above we used z to denote a generic point $(x, p) \in \mathbb{R}^6$. In the same vein, we sometimes write ζ for the generic wave variables (ψ, ϖ).

A *physical generalized state* of the regularized Vlasov model constitutes a complete specification of all its first-order evolutionary variables. We accordingly define the set Γ of all possible physical generalized states at time $t \in \mathbb{R}$ to be the subset of points

$$\mathfrak{Z}_t := (\mu_t, \zeta_t) \in M(\mathbb{R}^6) \oplus \dot{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \tag{63}$$

for which $\mu_t \in P(\mathbb{R}^6)$. The subset Γ_{B} of Γ denotes those physical generalized states which satisfy (59) and (60), and for which $\mathrm{supp}\,(\mu_0(\mathrm{d}x \times \mathbb{R}^3)) \subset B_R$.

In (63), the first direct sum is clearly in the sense of Banach spaces while the second may be either in Banach or Hilbert space sense (with the understanding then that Hilbert binds stronger than Banach on its left); since it is a little awkward to have two direct sum symbols with different meanings in a single expression, *for the Vlasov model* we use the Banach space meaning throughout.

In this paper we are only interested in systems with finite energy, momentum, and angular momentum (the mass of a system is finite by default, namely unity). Thus, for a suitable subset of *generic* physical generalized states $\mathfrak{Z}$ in Γ_{B} we formally define the mass functional

$$\mathcal{M}\,(\mathfrak{Z}) = \int \mu(\mathrm{d}z)\,, \tag{64}$$

the momentum functional

$$\mathcal{P}\,(\mathfrak{Z}) = \int p\,\mu(\mathrm{d}z) - \int \varpi \partial_x \psi \,\mathrm{d}x\,, \tag{65}$$

the angular momentum functional

$$\mathcal{J}\,(\mathfrak{Z}) = \int x \times p\,\mu(\mathrm{d}z) - \int \varpi\, x \times \partial_x \psi \,\mathrm{d}x\,, \tag{66}$$

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and the energy functional

$$\mathcal{E}\left(\mathfrak{Z}\right) = \int\left(\sqrt{1+|p|^2} + (\varrho * \psi)\right)\mu(\mathrm{d}z) + \frac{1}{2}\int\left(\left|\partial_x\psi\right|^2 + |\varpi|^2\right)\mathrm{d}x\,. \qquad (67)$$

Here, in keeping with our already stipulated abbreviations, $\mathrm{d}z$ denotes the Lebesgue measure $\mathrm{d}x\mathrm{d}p$ on $\mathbb{R}^6$ and $\int ...\mu(\mathrm{d}z)$ stands for $\iint ...\mu(\mathrm{d}x\mathrm{d}p)$. We restrict the set of physical generalized states to measures with finite expected values of $|x|$ and $|p|$. Now $|x|$ and $|p|$, understood as functions on $\mathbb{R}^6$, are not in $C_0^0(\mathbb{R}^6)$, but they are lower semi-continuous and therefore Radon measurable; hence, our condition of finite expected values of $|x|$ and $|p|$ (equivalently, of $|z|$) defines a proper subset $P_1(\mathbb{R}^6)$ of the Borel probability measures. The corresponding subset of the physical states Γ_B is denoted by $\Gamma_{\mathrm{B},1}$; the energy, momentum, and angular momentum are well-defined on $\Gamma_{\mathrm{B},1}$.

It remains to stipulate a suitable topology on $\Gamma_{\mathrm{B},1}$ in which the maps $t \mapsto \mathfrak{Z}_t$ for $t \in \mathbb{R}$ are continuous curves in $\Gamma_{\mathrm{B},1}$ that qualify as *generalized solutions* of (62) (given the fields). Unfortunately, the Banach space topology which $\Gamma_{\mathrm{B},1}$ naturally inherits as a subset of $C_0^0(\mathbb{R}^6)^* \oplus \dot{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ is too strong to study families of empirical atomic measures, for any two (atomic) empirical measures with disjoint supports are always at a distance 2 from each other in the metric induced by the TV topology.

A more suitable topology that immediately comes to mind is the *vague* (a.k.a. weak*) topology on $M(\mathbb{R}^6) \equiv C_0^0(\mathbb{R}^6)^*$ induced by $C_0^0(\mathbb{R}^6)$. The set $\Gamma_{\mathrm{B},1}$ with the vague topology on $M(\mathbb{R}^6)$ in place of the TV topology is denoted by $\Gamma_{\mathrm{B},1}^v$. However, since we are interested only in the subset $P_1(\mathbb{R}^6) \subset M(\mathbb{R}^6)$, we can do somewhat better and equip $P_1(\mathbb{R}^6)$ with the the standard Kantorovich–Rubinstein topology[7] induced by the dual Lipschitz distance in $P_1(\mathbb{R}^6)$ (a map on $P_1(\mathbb{R}^6) \times P_1(\mathbb{R}^6)$),

$$\mathrm{dist}_{\mathrm{L}^*}(\mu_1, \mu_2) := \sup_{g \in C^{0,1}(\mathbb{R}^6)}\left\{\left|\int g\,\mathrm{d}(\mu_1 - \mu_2)\right| : \mathrm{Lip}\,(g) \leq 1\right\}. \qquad (68)$$

We write $\mu_n \rightsquigarrow \mu$ if $\mathrm{dist}_{\mathrm{L}^*}(\mu_n, \mu) \to 0$.

Since it is convenient for the presentation to have a Banach space, we note that the metric $\mathrm{dist}_{\mathrm{L}^*}(\,.\,,\,.\,)$ defines a norm on $P_1 - P_1$ by $\|\sigma\|_{\mathrm{L}^*} := \mathrm{dist}_{\mathrm{L}^*}(\sigma_+, \sigma_-)$ for $\sigma \in (P_1 - P_1)(\mathbb{R}^6)$. As described in Appendix A.1, $\|\,.\,\|_{\mathrm{L}^*}$ can be extended[8] to a norm $\|\,.\,\|_{\widetilde{\mathrm{L}^*}}$ on the linear span of $P_1(\mathbb{R}^6)$, such that $\|\sigma\|_{\widetilde{\mathrm{L}^*}} = \|\sigma\|_{\mathrm{L}^*}$ whenever $\sigma(\mathbb{R}^6) = 0$. The completion of the linear span of $P_1(\mathbb{R}^6)$ w.r.t. $\|\,.\,\|_{\widetilde{\mathrm{L}^*}}$, denoted $\widetilde{M}_1(\mathbb{R}^6)$, is a Banach

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[7]The relationship between the various topologies is summarized in Appendix A.1.

[8]This extension will not be needed for any of our technical estimates.

space. We also write $\widetilde{P}_1(\mathbb{R}^6)$ for $P_1(\mathbb{R}^6) \hookrightarrow \widetilde{M}_1(\mathbb{R}^6)$. By $\widetilde{\Gamma}_{\mathrm{B},1}$ we denote the closed subset of generic phase space points

$$\mathbf{\mathfrak{z}} = (\mu; \zeta) \in \widetilde{M}_1(\mathbb{R}^6) \oplus (\dot{H}^1 \oplus L^2)(\mathbb{R}^3) \tag{69}$$

for which $\mu \in \widetilde{P}_1(\mathbb{R}^6)$, and for which ζ satisfies (59), (60); once again, the Banach space direct sum is meant on the right-hand side, so that in particular the norm of $\mathbf{\mathfrak{z}}_t$ reads

$$\|\mathbf{\mathfrak{z}}\| := \|\mu\|_{\widetilde{\mathrm{L}}^*} + \|\zeta\|_{HL} \tag{70}$$

with $\|\zeta_t\|_{HL} = \|\psi\|_{\dot{H}^1} + \|\varpi\|_{L^2}$.

3.2.1 Generalized solutions w.r.t. to the vague topology for μ

Considering first the field variables $\zeta \in C^1(\mathbb{R}, (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$ as given, we will call *M-vague solution* of (62) a C^1 map $t \mapsto \mu_t$ satisfying (62) with μ_t in place of f, for all $t \in \mathbb{R}$ integrated against any test function $g \in C_0^1(\mathbb{R}^6)$, and with the ∂_t pulled in front of the corresponding integral. Accordingly, a map $t \mapsto \mathbf{\mathfrak{z}}_t = (\mu_t; \zeta_t) \in C^1(\mathbb{R}, \Gamma_{\mathrm{B},1})$ will be called *M-vague HL-strong solution* of (57), (58), (62).

With the help of the concept of the M-vague HL-strong solution of (57), (58), (62), we can now immediately reformulate Theorem 2.12 into an existence result for what we call M-vague N-body solutions of the regularized Vlasov model (57), (58), (62).

THEOREM 3.2 *Let $t \mapsto (\mathbf{z}_t^{(N)}, \zeta_t^{(N)}) = \mathbf{\mathfrak{z}}_t^{(N)} \in C^1(\mathbb{R}, \Gamma_{\mathrm{B}}^{(N)})$, with $\lim_{t \to 0} \mathbf{\mathfrak{z}}_t^{(N)} = (\mathbf{z}_0^{(N)}, \zeta[\mathbf{z}_0^{(N)}])$, be the unique strong solution of the Hamiltonian field & N-body problem (16)–(20), and denote the empirical measure associated to $\mathbf{z}_t^{(N)}$ by*

$$\varepsilon[\mathbf{z}_t^{(N)}](\mathrm{d}x\mathrm{d}p) = \tfrac{1}{N}\textstyle\sum_{k=1}^N \delta_{q_k^{(N)}(t)}(\mathrm{d}x) \times \delta_{p_k^{(N)}(t)}(\mathrm{d}p) \tag{71}$$

Then $(t \mapsto (\varepsilon[\mathbf{z}_t^{(N)}]; \zeta_t^{(N)}) = \mathbf{\mathfrak{z}}_t^{(N)}) \in C^1(\mathbb{R}, \Gamma_{\mathrm{B},1}^v)$ is an M-vague N-body solution of the regularized wave gravity Vlasov equations (57), (58), (62), satisfying the Cauchy data $\lim_{t \to 0} \mathbf{\mathfrak{z}}_t^{(N)} = \mathbf{\mathfrak{z}}_0^{(N)}$, and conserving mass, momentum, angular momentum, and energy:

$$\mathcal{M}\left(\mathbf{\mathfrak{z}}_t^{(N)}\right) = M, \tag{72}$$

$$\mathcal{P}\left(\mathbf{\mathfrak{z}}_t^{(N)}\right) = P, \tag{73}$$

$$\mathcal{J}\left(\mathbf{\mathfrak{z}}_t^{(N)}\right) = J, \tag{74}$$

$$\mathcal{E}\left(\mathbf{\mathfrak{z}}_t^{(N)}\right) = E, \tag{75}$$

with M, P, J, E independent of time; in particular, $M = 1$.

Note that by Thm. 3.2 the set $\Gamma_{\mathrm{B}}^{(N)}$ becomes identified with a subset of $\Gamma_{\mathrm{B},1}^v$.

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3.2.2 Generalized solutions w.r.t. the Kantorovich–Rubinstein topology for μ

Since $\widetilde{\Gamma}_{\mathrm{B},1}$ is equipped with a Banach space topology, a map $t \mapsto \mathfrak{Z}_t = (\mu_t; \zeta_t)$ $\in C^1(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ satisfying (57), (58), (62) is properly called a $\widetilde{\Gamma}_{\mathrm{B},1}$-*strong generalized solution* of our regularized Vlasov model. Such solution satisfy the conventional conservation laws. Particular $\widetilde{\Gamma}_{\mathrm{B},1}$-strong generalized solutions, called $\widetilde{\Gamma}_{\mathrm{B},1}$-*strong N-body solutions*, are generated by the solutions of the field & N-body model of section 2. We summarize this in

THEOREM 3.3 *Let $t \mapsto \mathfrak{Z}_t = (\mu_t; \zeta_t) \in C^1(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ be a $\widetilde{\Gamma}_{\mathrm{B},1}$-strong generalized solution of (57), (58), (62) with Cauchy data $\lim_{t\to 0} \mathfrak{Z}_t = \mathfrak{Z}_0$. Then mass, momentum, angular momentum, and energy are conserved; i.e. (72), (73), (74), (75) hold. In particular, let $t \mapsto \mathfrak{z}_t^{(N)} \in C^1(\mathbb{R}, \Gamma_{\mathrm{B}}^{(N)})$ and $\varepsilon[\mathbf{z}_t^{(N)}](\mathrm{d}x\mathrm{d}p)$ be given as in Theorem 3.2. Then $t \mapsto \mathfrak{Z}_t^{(N)} = (\varepsilon[\mathbf{z}_t^{(N)}]; \zeta_t^{(N)}) \in C^1(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ is a $\widetilde{\Gamma}_{\mathrm{B},1}$-strong N-body solution of the regularized wave gravity Vlasov equations (57),(58),(62), with Cauchy data $\sup_{t\to 0} \mathfrak{Z}_t^{(N)} = \mathfrak{Z}_0^{(N)}$.*

Note that by Theorem 3.3 the set $\Gamma_{\mathrm{B}}^{(N)}$ becomes identified with a subset of $\widetilde{\Gamma}_{\mathrm{B},1}$.

We next show that arbitrary initial data $\mathfrak{Z}_0 \in \Gamma_{\mathrm{B},1}$ launch a unique $\widetilde{\Gamma}_{\mathrm{B},1}$-strong generalized solution $t \mapsto \mathfrak{Z}_t \in C^1(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ of our Vlasov model. Since the vague topology on P_1 is controlled by the standard Kantorovich–Rubinstein topology, solutions of the type $C^1(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ are automatically solutions of the type $C^1(\mathbb{R}, \Gamma_{\mathrm{B},1}^v)$.

3.3 The Cauchy problem for $\widetilde{\Gamma}_{\mathrm{B},1}$-strong solutions

To study the general Cauchy problem for (57), (58), (62) in the $\widetilde{\Gamma}_{\mathrm{B},1}$-strong topology, we rewrite (57), (58), (62) together with their Cauchy data as a fixed point problem,

$$\mathfrak{Z}_{\cdot} = F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\mathfrak{Z}_0), \tag{76}$$

where $F_{\cdot,0}$ is a continuous map from $C^0(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ into $C^0(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$, conditioned on $\mathfrak{Z}_0 \in \Gamma_{\mathrm{B},1}$. We will show that, w.r.t. a suitably weighted sup-norm, a truncated version of F is a Lipschitz map, with Lipschitz constant < 1, from a closed subset of weighted $C^0(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ into itself. Existence of a unique fixed point of the truncated F then follows from the standard contraction mapping theorem. By bootstrapping regularity, fixed points of the full F will then be shown to exist and to be in $C^1(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$, thus furnishing unique $\widetilde{\Gamma}_{\mathrm{B},1}$-strong solutions of (57), (58), (62) that conserve mass (64), momentum (65), angular momentum (66), and energy (67).

3.3.1 Definition of the fixed point map

Given any $\mathbf{3}_0 = (\mu_0; \zeta_0) \in \widetilde{\Gamma}_{\mathrm{B},1}$, for each t the map $F_{t,0}(.\,; . \,|\mu_0; \zeta_0)$ is given by

$$F_{t,0}(\mu.; \zeta. | \mu_0; \zeta_0) \equiv \left(\Pi^{\dagger}_{t,0}[\zeta.](\mu_0) \,; \, \Phi_{t,0}[\mu.](\zeta_0) \right) , \tag{77}$$

where $\Pi^{\dagger}_{t,0}[\zeta.](\mu_0) \equiv \mu_0 \circ \Pi_{0,t}[\zeta.]$, and where $\Pi_{.,.}$ and $\Phi_{.,.}$ are two-parameter groups, *flows* on the phase subspaces of the particles and the fields, respectively. *Given* a trajectory $\zeta.$ in field space, $\Pi_{.,.}[\zeta.]$ is the particle phase space flow, and *given* a trajectory $\mu.$ in probability measure space, $\Phi_{.,.}[\mu.]$ is the field phase space flow.

As to the flow on particle phase space, let $t \mapsto \zeta_t \in C^0_b(\mathbb{R}, (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$ be a generic, bounded continuous curve in $(\dot{H}^1 \oplus L^2)(\mathbb{R}^3)$. Given $t \mapsto \zeta_t$, the characteristic equations for (62) are the Hamiltonian equations for test particle motion $\mathrm{d}z/\mathrm{d}t = \mathbf{J} \cdot \partial_z \mathcal{H}(z, \zeta_t)$, with $z = (q, p)$, where $\mathbf{J}$ is the symplectic matrix, and $\mathcal{H}(z, \zeta_t)$ is the Hamiltonian (26) for $N = 1$ and with (z, ζ_t) substituted for $\mathbf{3}^{(1)}$. Explicitly,

$$\mathbf{J} \cdot \partial_z \mathcal{H}(z, \zeta_t) = \left(\partial_p \sqrt{1 + |p|^2} \,, \, -\partial_x \big(\varrho * \psi(\,.\,, t) \big)(q) \right) ; \tag{78}$$

note that only the ψ part of ζ enters in (78). The particle phase space flow $\Pi_{.,.}[\zeta.]$ is now defined implicitly as follows: given $t \mapsto \psi(\,.\,, t)$, for each solution $z. \in C^1(\mathbb{R}, \mathbb{R}^6)$ of the characteristic equations the integrated characteristic equations give the identity

$$z_t = z_{t'} + \int_{t'}^{t} \mathbf{J} \cdot \partial_z \mathcal{H}(z_\tau, \zeta_\tau) \mathrm{d}\tau =: \Pi_{t,t'}[\zeta.](z_{t'}) , \tag{79}$$

the r.h.s. of which being the transition function from some z at time t' to another z at time t for *all* t, t'; considering the totality of all t, t' gives the particle flow.

Similarly, to define the flow on field phase space, suppose $t \mapsto \mu_t \in C^0(\mathbb{R}, \widetilde{P}_1(\mathbb{R}^6))$ is given, and let $\rho_t(\mathrm{d}x) = \int \mu_t(\mathrm{d}x\mathrm{d}p)$, and $(\varrho * \rho_t)(x) = \int \varrho(y - x)\rho_t(\mathrm{d}y)$. Then $\frac{1}{2T} \int_{-T}^{T} \|\varrho * \rho_t\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}t \leq \|\varrho * \varrho\|_{\mathrm{u}} \left(\equiv \|\varrho\|^2_{L^2} \right)$. Given such $\varrho * \rho. \in L^2([-T, T], L^2(\mathbb{R}^3))$ for any $T > 0$, the solution $\zeta. = (\psi, \varpi)(\,.\,,.\,)$ to the wave equation with field source $\varrho * \rho.$ defines the flow $\Phi_{.,.}[\mu.]$ on field space through the transition function

$$\zeta_t =: \Phi_{t,t'}[\mu.](\zeta_{t'}) . \tag{80}$$

An explicit representation of (80) in terms of Fourier & Laplace transforms is available. However, by the higher regularity of $\varrho * \rho.$, the $\dot{H}^1$ and L^2 estimates of $\Phi_{.,.}[\mu.]$

are conveniently obtained from Kirchhoff's explicit pointwise expressions for classical solutions. In components, $\Phi_{\cdot,\cdot}[\mu_\cdot] \equiv (\Phi^\psi_{\cdot,\cdot}[\mu_\cdot], \Phi^\varpi_{\cdot,\cdot}[\mu_\cdot])$ reads ([Bre93], [Ika00], [ShSt00])

$$
\begin{aligned}
\psi(x,t) &= \fint_{\mathbb{S}^2} \Big([1 + (t-t')\Omega\cdot\nabla]\,\psi(x',t') + (t-t')\varpi(x',t') \\
&\qquad\qquad - \int_{t'}^t (t-t'')(\varrho * \rho_{t''})(x'')\mathrm{d}t'' \Big)\mathrm{d}\Omega \\
&=: \Phi^\psi_{t,t'}[\mu_\cdot](\zeta_{t'})(x) \tag{81} \\
\varpi(x,t) &= \fint_{\mathbb{S}^2} \Big([1 + 2(t-t')\Omega\cdot\nabla]\Omega\cdot\nabla\psi(x',t') + [1+(t-t')\Omega\cdot\nabla]\varpi(x',t') \\
&\qquad\qquad - \int_{t'}^t [1+(t-t'')\Omega\cdot\nabla](\varrho * \rho_{t''})(x'')\mathrm{d}t'' \Big)\mathrm{d}\Omega \\
&=: \Phi^\varpi_{t,t'}[\mu_\cdot](\zeta_{t'})(x)\,, \tag{82}
\end{aligned}
$$

where $x'^{\,\mathrm{or}\,''} = x + (t-t'^{\,\mathrm{or}\,''})\Omega$, where $\Omega \in \mathbb{S}^2$, and where $\fint_{\mathbb{S}^2}$ is short for $\frac{1}{4\pi}\int_{\mathbb{S}^2}$.

Having defined $\Pi_{\cdot,\cdot}[\zeta_\cdot]$ and $\Phi_{\cdot,\cdot}[\mu_\cdot]$, we are now ready to analyse equation (76).

3.3.2 Statement of the main fixed point results

So far, (76) has been defined purely formally as a rewriting of (57), (58), (62), with Cauchy data imposed. Our first duty should be to show that (76) in fact makes sense, viz. that F maps a relevant, closed subset of $C^0(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ into itself, indeed. We prove this as a byproduct of the auxiliary result that a truncated version of $F_{\cdot,0}(\,.\,|\mathfrak{Z}_0)$ is a Lipschitz map, with Lipschitz constant < 1, from a closed subset of $C^0(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ into itself, where "closed" is meant w.r.t. (a suitably weighted) sup-norm.

We note that by a density argument for the curves of empirical measures $t \mapsto \rho_t^{(N)}$, the a-priori estimates of section 2.3.1 extend to our regularized Vlasov model. Hence, any Vlasov solution $t \mapsto \mathfrak{Z}_t$ must be in some subset of $C^0(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ satisfying $\|\mathfrak{Z}_t\| \leq c_0 + c_1|t| + c_2 t^2$ for some positive constants c_0, c_1 and c_2. This suggests to work with the closure of the bounded continuous functions from $\mathbb{R}$ to $\widetilde{\Gamma}_{\mathrm{B},1}$, denoted $C^0_b(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$, w.r.t. a *weighted sup-norm* of $\mathfrak{Z}_\cdot$ given by $\sup_{t\in\mathbb{R}}\left([c_0 + c_1|t| + c_2 t^2]^{-1}\|\mathfrak{Z}_t\|\right)$; however, for technical reasons it is more convenient to close $C^0_b(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ w.r.t. the weighted sup-norm

$$
\|\mathfrak{Z}_\cdot\|_{\mathrm{w}} = \sup_{t\in\mathbb{R}}\left(e^{-w|t|}\|\mathfrak{Z}_t\|\right) \tag{83}
$$

for some $w > 0$; eventually we will restrict w to $w > \underline{w} > 0$. The closure of $C^0_b(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ w.r.t. norm (83) is a Banach space, denoted $C^0_{\mathrm{w}}(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$; the subscript w can be read as

meaning both "weighted" and reference to the parameter w in the definition (83). We also introduce the Banach space $C_w^0(\mathbb{R}_+, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$ with norm $\|\mathbf{3}.\|_w = \sup_{t \geq 0} (e^{-wt} \|\mathbf{3}_t\|)$. By the time reversal symmetry of (76) it suffices to limit the discussion to $t \geq 0$.

Moreover, as regards the argument $\mathbf{3}.(= (\mu_.; \zeta_.))$ of $F_{.,0}(\,.\,|\mathbf{3}_0)(= F_{.,0}(\,.\,;\,.\,|\mu_0; \zeta_0))$ given in (77), it is not a-priori required that $\lim_{t\to 0} \mathbf{3}_t$ of $\mathbf{3}.$ coincides with the given $\mathbf{3}_0$; in fact, it is not even a-priori necessary that the measure component of $\mathbf{3}.$ is in P_1 but could as well be in $\widetilde{M_1}$. However, for the solution of (76) this must be so, for $F_{.,0}(\,.\,;\,.\,|\mu_0; \zeta_0)$ has been constructed such that $F_{0,0}(\mu_.; \zeta_.|\mu_0; \zeta_0) = (\mu_0; \zeta_0) = \mathbf{3}_0$, as is readily verified by inspection of $\Pi_{0,0}$ and $\Phi_{0,0}$. Therefore, with the exception of some technical estimates that we will highlight explicitly, we only need to apply $F_{.,0}(\,.\,|\mathbf{3}_0)$ to those $\mathbf{3}. \in C^0(\mathbb{R}, \widetilde{\Gamma}_{B,1})$ which satisfy $\lim_{t\to 0} \mathbf{3}_t = \mathbf{3}_0$. We denote the corresponding subsets of $C_w^0(\mathbb{R}, \widetilde{\Gamma}_{B,1})$ and $C_w^0(\mathbb{R}_+, \widetilde{\Gamma}_{B,1})$ by $C_w^0(\mathbb{R}, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$ and $C_w^0(\mathbb{R}_+, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$, respectively. Furthermore we denote the free evolution of the initial data $\mathbf{3}_0$ by $\mathbf{3}_.^0 := F_{.,0}(0_.|\mathbf{3}_0) \in C_w^0(\mathbb{R}, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$; here, $0_.$ is the trivial constant map $(t \mapsto 0) \in C_w^0\left(\mathbb{R}, \widetilde{M_1}(\mathbb{R}^6) \oplus \dot{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\right)$. We will work with certain closed subsets of $C_w^0(\mathbb{R}_{(+)}, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$. By $B_{\widetilde{R}}(\mathbf{3}_.^0) \subset C_w^0(\mathbb{R}_{(+)}, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$ we denote a closed ball of radius $\widetilde{R}$ centered at $\mathbf{3}_.^0$ (with $t \in \mathbb{R}$ or $\mathbb{R}_+$). Furthermore, we shall need the closed subsets defined by the condition $\sup_{t\geq 0} \|\psi(\,.\,,t)\|_{\dot{H}^1} \leq C_\psi$ with $C_\psi \geq \|\psi_0\|_{\dot{H}^1}$. In this vein, we also introduce a truncation of $F_{.,0}(\,.\,|\mathbf{3}_0)$, denoted $\overline{F}_{.,0}(\,.\,|\mathbf{3}_0)$, which for each $t > 0$ is obtained from $F_{t,0}(\,.\,|\mathbf{3}_0)$ by replacing $\Phi_{t,0}^\psi[\mu_.](\zeta_0)$ by

$$\overline{\Phi}_{t,0}^\psi[\mu_.](\zeta_0) := \min\left\{1, C_\psi \|\Phi_{t,0}^\psi[\mu_.](\zeta_0)\|_{\dot{H}^1}^{-1}\right\} \Phi_{t,0}^\psi[\mu_.](\zeta_0) \tag{84}$$

PROPOSITION 3.4 *For every $\mathbf{3}_0 \in \Gamma_{B,1}$, there exist $\underline{C}_\psi \geq \|\psi_0\|_{\dot{H}^1}$ and $\underline{w} > 0$, such that $\overline{F}_{.,0}(\,.\,|\mathbf{3}_0)$ is a Lipschitz map with Lipschitz constant < 1 which maps the closed subsets of balls $B_{\widetilde{R}}(\mathbf{3}_.^0) \subset C_w^0(\mathbb{R}_+, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$ for which $\sup_{t\geq 0} \|\psi(\,.\,,t)\|_{\dot{H}^1} \leq C_\psi$ into themselves whenever $\widetilde{R} \geq \|\mathbf{3}_.^0\|_w$, $w > \underline{w}$, and $C_\psi \geq \underline{C}_\psi \geq \|\psi_0\|_{\dot{H}^1}$.*

By the standard contraction mapping theorem, an immediate corollary to Proposition 3.4 is the existence of a unique fixed point $(t \mapsto \mathbf{3}_t) \in C_w^0(\mathbb{R}_+, \widetilde{\Gamma}_{B,1})$, with $(t \mapsto \psi(\,.\,,t)) \in C_b^0(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$, of the fixed point equation with the truncated F,

$$\mathbf{3}. = \overline{F}_{.,0}(\mathbf{3}.|\mathbf{3}_0). \tag{85}$$

By bootstrapping regularity, fixed points of the untruncated F will then be shown to exist and to actually be in $C^1(\mathbb{R}, \widetilde{\Gamma}_{B,1})$, furnishing unique $\widetilde{\Gamma}_{B,1}$-strong Vlasov solutions. Thus we may state our main existence and uniqueness theorem of this section.

THEOREM 3.5 *For every $\mathfrak{Z}_0 \in \Gamma_{\mathrm{B},1}$ there exists $\underline{w} > 0$ such that whenever $w > \underline{w}$, the Vlasov fixed point equation (76) with Cauchy data $\lim_{t \to 0} \mathfrak{Z}_t = \mathfrak{Z}_0$ is solved by a unique curve $t \mapsto \mathfrak{Z}_t \in C^0_{\mathrm{w}}(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$; since also $\psi_0 \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3)$, the map $t \mapsto \mathfrak{Z}_t \in (C^0_{\mathrm{w}} \cap C^1)(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$, and thus it is the unique $\widetilde{\Gamma}_{\mathrm{B},1}$-strong solution to (57), (58), (62) conserving mass (64), momentum (65), angular momentum (66), and energy (67).*

3.3.3 Proof of Proposition 3.4

We begin with auxiliary results concerning the flow on the particle sub-phase space.

LEMMA 3.6 *Given any curve $\zeta_. \in C^k(\mathbb{R}, (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$, $k = 0, 1, ...$, we have*
(i) $\mathbf{J} \cdot \partial_z \mathcal{H}(\,.\,, \zeta_.) \in C^k(\mathbb{R} \times \mathbb{R}^6, \mathbb{R}^6)$,
(ii) $\mathbf{J} \cdot \partial_z \mathcal{H}(\,.\,, \zeta_t) \in C^\infty(\mathbb{R}^6, \mathbb{R}^6)$;
(iii) $\partial_z \cdot \mathbf{J} \cdot \partial_z \mathcal{H}(\,.\,, \zeta_t) \equiv 0$;
(iv) $|\mathbf{J} \cdot \partial_z \mathcal{H}(\,.\,, \zeta_t)| \leq 1 + \|\varrho\|_{L^2} \|\psi(\,.\,, t)\|_{\dot{H}^1}$.

Proof of Lemma 3.6: Regularity (i), (ii), and incompressibility (iii), are obvious. The bound (iv) obtains by using the triangle inequality, then $|p| \leq \sqrt{1 + |p|^2}$ for the momentum part, respectively for the space part the Cauchy–Schwarz inequality to get $|\varrho * \nabla \psi|(x) \leq \|\varrho\|_{L^2} \|\psi\|_{\dot{H}^1}$ for all x; cf. (37).   Q.E.D.

As a straightforward spin-off of Lemma 3.6, we have

COROLLARY 3.7 *If $\zeta_. \in C^k(\mathbb{R}, (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$, then $\Pi_{.,.}[\zeta_.] \in C^k(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^6, \mathbb{R}^6)$, and $\Pi_{t,t'}[\zeta_.]$ is a symplectomorphism $\forall t, t' \in \mathbb{R}$; in particular, $\det \partial_z \Pi_{t,t'}[\zeta_.](z) = 1$.*

Proof of Corollary 3.7: This is a standard corollary. See, e.g. [HiSm74].   Q.E.D.

Controlling the field space component of $\overline{F}_{.,0}$ requires only the following Lemma:

LEMMA 3.8 *If $\mu_. \in C^0(\mathbb{R}, \widetilde{P}_1)$, then $\Phi_{.,.}[\mu_.] \in C^0(\mathbb{R} \times \mathbb{R} \times (\dot{H}^1 \oplus L^2)(\mathbb{R}^3), (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$.*

Proof of Lemma 3.8: For ζ_0 classical: straightforward calculation, for (81), (82) are quite explicit. Then apply the Hahn-Banach theorem.Q.E.D.

Proof of Proposition 3.4: We first show that, given any $\mathfrak{Z}_0 \in \Gamma_{\mathrm{B},1}$, the map $F_{.,0}(\,.\,|\mathfrak{Z}_0)$ is Lipschitz-continuous from a closed subset of $C^0_{\mathrm{w}}(\mathbb{R}_+, \widetilde{\Gamma}_{\mathrm{B},1}|\mathfrak{Z}_0)$, defined by the condition $\sup_{t \geq 0} \|\psi(\,.\,, t)\|_{\dot{H}^1} \leq C_\psi$ with $C_\psi \geq \underline{C}_\psi \geq \|\psi_0\|_{\dot{H}^1}$, to $C^0_{\mathrm{w}}(\mathbb{R}_+, \widetilde{\Gamma}_{\mathrm{B},1}|\mathfrak{Z}_0)$ whenever $w > \underline{w}$, with $\underline{w}$ depending at most on ϱ, C_ψ, and the Lipschitz constant at most on ϱ, C_ψ, w.

We emphasize that the conditioning $\lim_{t\to 0} \mathbf{3}_t = \mathbf{3}_0$ and $\lim_{t\to 0} \widetilde{\mathbf{3}}_t = \mathbf{3}_0$ implied by the definition of $C^0_{\mathrm{w}}(\mathbb{R}_+, \widetilde{\Gamma}_{\mathrm{B},1}|\mathbf{3}_0)$ do not enter our estimates.

To break up the proof into two parts, we use the triangle inequality in the form

$$
\begin{aligned}
\|F_{\cdot,0}(\mu_{\cdot}; \zeta_{\cdot}|\mathbf{3}_0) - F_{\cdot,0}(\tilde{\mu}_{\cdot}; \tilde{\zeta}_{\cdot}|\mathbf{3}_0)\|_{\mathrm{w}} &\leq \|F_{\cdot,0}(\mu_{\cdot}; \zeta_{\cdot}|\mathbf{3}_0) - F_{\cdot,0}(\tilde{\mu}_{\cdot}; \zeta_{\cdot}|\mathbf{3}_0)\|_{\mathrm{w}} \\
&+ \|F_{\cdot,0}(\tilde{\mu}_{\cdot}; \zeta_{\cdot}|\mathbf{3}_0) - F_{\cdot,0}(\tilde{\mu}_{\cdot}; \tilde{\zeta}_{\cdot}|\mathbf{3}_0)\|_{\mathrm{w}}.
\end{aligned} \tag{86}
$$

Given $\mathbf{3}_0 = (\mu_0, \zeta_0)$ and $C_\psi \geq \|\psi_0\|_{\dot{H}^1}$, we show that, (a) given $\zeta_{\cdot} \in C^0(\mathbb{R}_+, (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$ satisfying $\|\psi_t\|_{\dot{H}^1} \leq C_\psi$ for all $t > 0$, for any two $\mu_{\cdot}$ and $\tilde{\mu}_{\cdot}$ in $C^0(\mathbb{R}_+, \widetilde{P}_1)$ and all $w > 0$ we have

$$
\|F_{\cdot,0}(\mu_{\cdot}; \zeta_{\cdot}|\mathbf{3}_0) - F_{\cdot,0}(\tilde{\mu}_{\cdot}; \zeta_{\cdot}|\mathbf{3}_0)\|_{\mathrm{w}} \leq L_1[\varrho; w] \sup_{t\geq 0} \left(e^{-wt} \|\mu_t - \tilde{\mu}_t\|_{\mathrm{L}^*} \right), \tag{87}
$$

and (b), given $\mu_{\cdot} \in C^0(\mathbb{R}_+, \widetilde{P}_1)$, for any two $\zeta_{\cdot}$ and $\tilde{\zeta}_{\cdot}$ in $C^0(\mathbb{R}_+, (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$, satisfying $\max\{\|\psi_t\|_{\dot{H}^1}, \|\tilde{\psi}_t\|_{\dot{H}^1}\} \leq C_\psi$ for all $t > 0$, and for all $w > \underline{w}[\varrho; C_\psi]$ we have

$$
\|F_{\cdot,0}(\mu_{\cdot}; \zeta_{\cdot}|\mathbf{3}_0) - F_{\cdot,0}(\mu_{\cdot}; \tilde{\zeta}_{\cdot}|\mathbf{3}_0)\|_{\mathrm{w}} \leq L_2[\varrho; w, \underline{w}] \sup_{t\geq 0} \left(e^{-wt} \|\zeta_t - \tilde{\zeta}_t\|_{HL} \right); \tag{88}
$$

for then it follows from (86), (87), (88) that, given any $\mathbf{3}_0$ and $C_\psi \geq \underline{C}_\psi \geq \|\psi_0\|_{\dot{H}^1}$,

$$
\|F_{\cdot,0}(\mathbf{3}_{\cdot}|\mathbf{3}_0) - F_{\cdot,0}(\widetilde{\mathbf{3}}_{\cdot}|\mathbf{3}_0)\|_{\mathrm{w}} \leq L[\varrho; w, \underline{w}] \|\mathbf{3}_{\cdot} - \widetilde{\mathbf{3}}_{\cdot}\|_{\mathrm{w}} \tag{89}
$$

whenever $w > \underline{w}[\varrho, C_\psi]$, with $L[\varrho; w, \underline{w}] := \max\{L_1[\varrho; w], L_2[\varrho; w, \underline{w}]\}$.

Part a) To prove (87), we fix $\mathbf{3}_0$ and $\zeta_{\cdot}$ and note that in this case

$$
\|F_{t,0}(\mu_{\cdot}; \zeta_{\cdot}|\mathbf{3}_0) - F_{t,0}(\tilde{\mu}_{\cdot}; \zeta_{\cdot}|\mathbf{3}_0)\| = \|\Phi_{t,0}[\mu_{\cdot}](\zeta_0) - \Phi_{t,0}[\tilde{\mu}_{\cdot}](\zeta_0)\|_{HL}, \tag{90}
$$

where, in components,

$$
\begin{aligned}
\|\Phi_{t,0}[\mu_{\cdot}](\zeta_0) - \Phi_{t,0}[\tilde{\mu}_{\cdot}](\zeta_0)\|_{HL} &= \|\Phi_{t,0}^{\psi}[\mu_{\cdot}](\zeta_0) - \Phi_{t,0}^{\psi}[\tilde{\mu}_{\cdot}](\zeta_0)\|_{\dot{H}^1} \\
&+ \|\Phi_{t,0}^{\varpi}[\mu_{\cdot}](\zeta_0) - \Phi_{t,0}^{\varpi}[\tilde{\mu}_{\cdot}](\zeta_0)\|_{L^2}.
\end{aligned} \tag{91}
$$

Furthermore, using (81) and then the definition of $\|\cdot\|_{\dot{H}^1}$, we have

$$
\|\Phi_{t,0}^{\psi}[\mu_{\cdot}](\zeta_0) - \Phi_{t,0}^{\psi}[\tilde{\mu}_{\cdot}](\zeta_0)\|_{\dot{H}^1}^2 = \int \left| \int_0^t \fint_{\mathbb{S}^2} (t - t') \nabla[\varrho * (\rho_{t'} - \tilde{\rho}_{t'})](x') \mathrm{d}\Omega \mathrm{d}t' \right|^2 \mathrm{d}x, \tag{92}
$$

while with (82) and the definition of $\|\cdot\|_{L^2}$, we find

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$$\left\| \Phi_{t,0}^{\varpi}[\mu_\cdot](\zeta_0) - \Phi_{t,0}^{\varpi}[\tilde\mu_\cdot](\zeta_0) \right\|_{L^2}^2$$

$$= \int \left| \int_0^t \fint_{\mathbb{S}^2} (1 + (t-t')\Omega \cdot \nabla)\, [\varrho * (\rho_{t'} - \tilde\rho_{t'})](x')\mathrm{d}\Omega \mathrm{d}t' \right|^2 \mathrm{d}x. \quad (93)$$

As to (92), triangle and Jensen's inequalities, and Fubini, yield the estimate

$$\left\| \Phi_{t,0}^{\psi}[\mu_\cdot](\zeta_0) - \Phi_{t,0}^{\psi}[\tilde\mu_\cdot](\zeta_0) \right\|_{\dot H^1}^2 \le \fint_{\mathbb{S}^2} \int \left[\int_0^t (t-t')\, \left|\nabla \left[\varrho * (\rho_{t'} - \tilde\rho_{t'}) \right](x')\right| \mathrm{d}t' \right]^2 \mathrm{d}x\mathrm{d}\Omega.$$
$$(94)$$

Now multiply (94) by e^{-2wt}, pull e^{-2wt} under the square in rhs(94) and note that

$$\int \left[e^{-wt} \int_0^t (t-t')\, \left|\nabla \left[\varrho * (\rho_{t'} - \tilde\rho_{t'}) \right](x')\right| \mathrm{d}t' \right]^2 \mathrm{d}x$$

$$= \int \left[\int_0^t \left(e^{-\frac12 w(t-t')}(t-t') \right) \left(e^{-\frac12 w(t-t')} e^{-wt'} \left|\nabla \left[\varrho * (\rho_{t'} - \tilde\rho_{t'}) \right](x')\right| \right) \mathrm{d}t' \right]^2 \mathrm{d}x$$

$$\le \int_0^t e^{-w(t-t'')}(t-t'')^2 \mathrm{d}t'' \int_0^t e^{-w(t-t')} e^{-2wt'} \int \left|\nabla \left[\varrho * (\rho_{t'} - \tilde\rho_{t'}) \right](x')\right|^2 \mathrm{d}x\mathrm{d}t'$$

$$\le \tfrac{2}{w^4} \sup_{t'\ge 0} \left(e^{-2wt'} \int \left|\nabla \left[\varrho * (\rho_{t'} - \tilde\rho_{t'}) \right](x')\right|^2 \mathrm{d}x \right), \quad (95)$$

the first inequality by Cauchy–Schwarz, followed by Fubini, the second inequality by Hölder followed by $\int_0^t e^{-w(t-t'')}(t-t'')^2 \mathrm{d}t'' \int_0^t e^{-w(t-t')}\mathrm{d}t' \le \int_0^\infty e^{-w\tau}\tau^2 \mathrm{d}\tau \int_0^\infty e^{-w\tau}\mathrm{d}\tau = 2/w^4$. We next estimate the remaining $\mathrm{d}x$ integral by itself. For this we first rewrite it with the help of one of Green's identities, a change of integration variables $x \to x'$, and Fubini's theorem, exchanging the $\mathrm{d}x'$ integration with one of the convolution integrations ($\mathrm{d}y$, say); we then apply the Kantorovich–Rubinstein duality twice to obtain generalized Hölder estimates, then use the estimate $\|\rho - \tilde\rho\|_{L^*} \le \|\mu - \tilde\mu\|_{L^*}$ for $\rho(\mathrm{d}x) = \mu(\mathrm{d}x \times \mathbb{R}^3)$ (similarly for $\tilde\rho$). Thus, independently of Ω, we have

$$\int \left|\nabla \left[\varrho * (\rho_{t'} - \tilde\rho_{t'}) \right](x')\right|^2 \mathrm{d}x = -\int \left(\varrho * \nabla^2 \varrho * (\rho_{t'} - \tilde\rho_{t'}) \right)(y)\, (\rho_{t'} - \tilde\rho_{t'})(\mathrm{d}y)$$

$$\le \mathrm{Lip}\left(\varrho * \nabla^2 \varrho * (\rho_{t'} - \tilde\rho_{t'}) \right)\|\rho_{t'} - \tilde\rho_{t'}\|_{L^*}$$

$$\le \mathrm{Lip}^{\circ 2}\!\left(\varrho * \nabla^2 \varrho \right)\|\rho_{t'} - \tilde\rho_{t'}\|_{L^*}^2$$

$$\le \mathrm{Lip}^{\circ 2}\!\left(\varrho * \nabla^2 \varrho \right)\|\mu_{t'} - \tilde\mu_{t'}\|_{L^*}^2, \quad (96)$$

where $\mathrm{Lip}^{\circ 2}(\varrho * \nabla^2 \varrho)$ is the *iterated Lipschitz constant*[9] of $\varrho * \nabla^2 \varrho$. We estimate rhs(95) with the help of (96), which in turn estimates $(e^{-2wt}\mathrm{rhs}(94))$ in such a way that the integration over $\mathrm{d}\Omega$ now factors out, yielding the factor unity. Thus, taking $\sup_{t \geq 0}(e^{-2wt}\,\mathrm{l.h.s.}(94))$ and then square roots yields

$$\sup_{t \geq 0}\left(e^{-wt}\|\Phi_{t,0}^{\psi}[\mu.](\zeta_0) - \Phi_{t,0}^{\psi}[\tilde{\mu}.](\zeta_0)\|_{\dot{H}^1}\right) \leq \sqrt{\mathrm{Lip}^{\circ 2}(\varrho * \nabla^2 \varrho)\tfrac{2}{w^4}} \sup_{t \geq 0}\left(e^{-wt}\|\mu_t - \tilde{\mu}_t\|_{\mathrm{L}^*}\right).$$
(97)

As for (93), we proceed similarly, except that after the Cauchy-Schwarz and Fubini steps we now use also that

$$\int |(1 + (t - t')\Omega \cdot \nabla)\, g(x)|^2 \mathrm{d}x = \int \left(g(x)\left(1 - (t-t')^2(\Omega \cdot \nabla)^2\right)g(x)\right)\mathrm{d}x$$
(98)

where $g(x) = [\varrho * (\rho_{t'} - \tilde{\rho}_{t'})](x')$, and obtain

$$\sup_{t \geq 0}\left(e^{-wt}\|\Phi_{t,0}^{\varpi}[\mu.](\zeta_0) - \Phi_{t,0}^{\varpi}[\tilde{\mu}.](\zeta_0)\|_{L^2}\right) \leq$$
$$\sqrt{\mathrm{Lip}^{\circ 2}(\varrho * \varrho)\tfrac{1}{w^2} + \mathrm{Lip}^{\circ 2}(\varrho * (\Omega_0 \cdot \nabla)^2 \varrho)\tfrac{2}{w^4}} \sup_{t \geq 0}\left(e^{-wt}\|\mu_t - \tilde{\mu}_t\|_{\mathrm{L}^*}\right), \quad (99)$$

where $\Omega_0 \in \mathbb{S}^2$ is arbitrary.

We now recall (91). Noting that, by triangle inequality, $\sup_{t \geq 0}\left(e^{-wt}\,\mathrm{l.h.s}(91)\right)$ is not bigger than the sum of (97) and (99), we arrive at

$$\sup_{t \geq 0}\left(e^{-wt}\|\Phi_{t,0}[\mu.](\zeta_0) - \Phi_{t,0}[\tilde{\mu}.](\zeta_0)\|_{HL}\right) \leq L_1[\varrho; w] \sup_{t \geq 0}\left(e^{-wt}\|\mu_t - \tilde{\mu}_t\|_{\mathrm{L}^*}\right), \quad (100)$$

with

$$L_1[\varrho; w] = \sqrt{\mathrm{Lip}^{\circ 2}(\varrho * \nabla^2 \varrho)\tfrac{2}{w^4}} + \sqrt{\mathrm{Lip}^{\circ 2}(\varrho * \varrho)\tfrac{1}{w^2} + \mathrm{Lip}^{\circ 2}(\varrho * (\Omega_0 \cdot \nabla)^2 \varrho)\tfrac{2}{w^4}}. \quad (101)$$

Finally recalling (90), we see that our proof of (87) is concluded.

---

[9]The iterated Lipschitz constant of f is given by

$$\mathrm{Lip}^{\circ 2}(f) = \sup_{x \neq y} \sup_{\tilde{x} \neq \tilde{y}} \frac{|f(x - \tilde{x}) + f(y - \tilde{y}) - f(x - \tilde{y}) - f(y - \tilde{x})|}{|x - y||\tilde{x} - \tilde{y}|}.$$

If $f \in C^2(\mathbb{R}^d)$, then $\mathrm{Lip}^{\circ 2}(f) = \sup_{x \in \mathbb{R}^d} \|\nabla^{\otimes 2} f(x)\|_\infty$, where $\nabla^{\otimes 2} f(x)$ is the Hessian of f at x and $\|M\|_\infty$ the sup norm (i.e. spectral radius) of a real symmetric matrix M.

Part b) To prove (88), we fix $\mathbf{3}_0$, and $\mu_\cdot$ and note that in this case

$$\|F_{t,0}(\mu_\cdot; \zeta_\cdot|\mathbf{3}_0) - F_{t,0}(\mu_\cdot; \tilde{\zeta}_\cdot|\mathbf{3}_0)\| = \|\mu_0 \circ \Pi_{0,t}[\zeta_\cdot] - \mu_0 \circ \Pi_{0,t}[\tilde{\zeta}_\cdot]\|_{\mathrm{L}^*}. \tag{102}$$

Recalling (68) and Corollary 3.7, we note next that

$$\|\mu_0 \circ \Pi_{0,t}[\zeta_\cdot] - \mu_0 \circ \Pi_{0,t}[\tilde{\zeta}_\cdot]\|_{\mathrm{L}^*} = \sup_{\substack{g \in C^{0,1}(\mathbb{R}^6) \\ \mathrm{Lip}(g) \leq 1}} \left| \int g \, \mathrm{d}(\mu_0 \circ \Pi_{0,t}[\zeta_\cdot] - \mu_0 \circ \Pi_{0,t}[\tilde{\zeta}_\cdot]) \right|$$

$$= \sup_{\substack{g \in C^{0,1}(\mathbb{R}^6) \\ \mathrm{Lip}(g) \leq 1}} \left| \int (g \circ \Pi_{t,0}[\zeta_\cdot] - g \circ \Pi_{t,0}[\tilde{\zeta}_\cdot]) \mathrm{d}\mu_0 \right|. \tag{103}$$

Pulling $|\,.\,|$ under the last integral in (103) and using $\mathrm{Lip}\,(g) \leq 1$ gives the estimate

$$\|\mu_0 \circ \Pi_{0,t}[\zeta_\cdot] - \mu_0 \circ \Pi_{0,t}[\tilde{\zeta}_\cdot]\|_{\mathrm{L}^*} \leq \int |\Pi_{t,0}[\zeta_\cdot](z) - \Pi_{t,0}[\tilde{\zeta}_\cdot](z)| \mu_0(\mathrm{d}z). \tag{104}$$

By (79), for $z_\cdot, \tilde{z}_\cdot \in C^1(\mathbb{R}, \mathbb{R}^6)$ solving the characteristic equations for given $t \mapsto \psi(\,.\,,t)$ and $t \mapsto \tilde{\psi}(\,.\,,t)$, respectively, with[10] $z_0 = z = \tilde{z}_0$, we have

$$\Pi_{t,0}[\zeta_\cdot](z) - \Pi_{t,0}[\tilde{\zeta}_\cdot](z) = \int_0^t \left(\mathbf{J} \cdot \partial_z \mathcal{H}(z_\tau, \zeta_\tau) - \mathbf{J} \cdot \partial_z \mathcal{H}(\tilde{z}_\tau, \tilde{\zeta}_\tau) \right) \mathrm{d}\tau \tag{105}$$

We now insert (105) in the right-hand side of (104), estimate the resulting expression by pulling the absolute bars under the t-integral and applying the triangle inequality, next simplify by noting that $\mathbf{J}$ is an isometry on $\mathbb{R}^6$, and use Fubini's theorem to exchange $\mathrm{d}\tau$ and $\mathrm{d}\mu_0$ integrations. Thus we obtain the estimate

$$\|\mu_0 \circ \Pi_{0,t}[\zeta_\cdot] - \mu_0 \circ \Pi_{0,t}[\tilde{\zeta}_\cdot]\|_{\mathrm{L}^*} \leq \int_0^t \int \left| \partial_z \mathcal{H}(z_\tau, \zeta_\tau) - \partial_z \mathcal{H}(z_\tau, \tilde{\zeta}_\tau) \right| \mu_0(\mathrm{d}z) \mathrm{d}\tau$$

$$+ \int_0^t \int \left| \partial_z \mathcal{H}(z_\tau, \tilde{\zeta}_\tau) - \partial_z \mathcal{H}(\tilde{z}_\tau, \tilde{\zeta}_\tau) \right| \mu_0(\mathrm{d}z) \mathrm{d}\tau. \tag{106}$$

Since we want an estimate for $\sup_{t \geq 0} (e^{-wt} \, \mathrm{l.h.s.}(106))$, we next consider the exponentially weighted suprema of the two integrals on r.h.s.(106) separately.

The first integral on r.h.s.(106) is estimated as follows. By (78) with $z_\tau = (q_\tau, p_\tau)$ and by the Cauchy–Schwarz inequality, we have

$$|\partial_z \mathcal{H}(z_\tau, \zeta_\tau) - \partial_z \mathcal{H}(z_\tau, \tilde{\zeta}_\tau)| = \left| \left(\varrho * \left[\nabla \psi(\,.\,,\tau) - \nabla \tilde{\psi}(\,.\,,\tau) \right] \right)(q_\tau) \right|$$

$$\leq \|\varrho\|_{L^2} \|\psi(\,.\,,\tau) - \tilde{\psi}(\,.\,,\tau)\|_{\dot{H}^1}. \tag{107}$$

---

[10]The initial data condition $z_0 = \tilde{z}_0$ derives from the $\mathbf{3}_0$ in $F_{\cdot,0}(\,.\,; \,.\,|\mathbf{3}_0)$.

Since r.h.s.(107) is independent of z, integration w.r.t. $\mathrm{d}\mu_0$ factors out and yields 1. Proceeding now similarly as in estimate (95), we obtain

$$\sup_{t\geq 0}\left(e^{-wt}\int_0^t \|\psi(\,.\,,\tau)-\tilde{\psi}(\,.\,,\tau)\|_{\dot{H}^1}\mathrm{d}\tau\right) \leq \tfrac{1}{w}\sup_{t\geq 0}\left(e^{-wt}\|\psi(\,.\,,t)-\tilde{\psi}(\,.\,,t)\|_{\dot{H}^1}\right). \quad (108)$$

Multiplying (108) by $\|\varrho\|_{L^2}$ gives an upper bound for $\sup_{t\geq 0}\left(e^{-wt}\int_0^t\int \text{r.h.s.}(107)\mathrm{d}\mu_0\mathrm{d}\tau\right)$, which in turn is an upper bound for $\sup_{t\geq 0}(e^{-wt}$ first integral on r.h.s.(106)).

As to the second integral on r.h.s.(106), its integrand is rewritten using (78), with $z_\tau = (q_\tau,p_\tau)$ and $\tilde{z}_\tau = (\tilde{q}_\tau,\tilde{p}_\tau)$, then estimated by the triangle inequality, giving

$$|\partial_z\mathcal{H}(z_\tau,\tilde{\zeta}_\tau) - \partial_z\mathcal{H}(\tilde{z}_\tau,\tilde{\zeta}_\tau)| \leq \left|\left(\varrho*\nabla\tilde{\psi}(\,.\,,\tau)\right)(q_\tau) - \left(\varrho*\nabla\tilde{\psi}(\,.\,,\tau)\right)(\tilde{q}_\tau)\right|$$
$$+\left|\frac{p_\tau}{\sqrt{1+|p_\tau|^2}} - \frac{\tilde{p}_\tau}{\sqrt{1+|\tilde{p}_\tau|^2}}\right|. \quad (109)$$

The two expressions on the right-hand side of (109) are now estimated separately.

The first term on r.h.s.(109) is estimated as follows. Spelling out the convolutions and factoring out $\nabla\tilde{\psi}$ in the integrand, then pulling $|\,.\,|$ into the convolution integral, then using the Cauchy–Schwarz inequality, we get

$$|(\varrho*\nabla\tilde{\psi}(\,.\,,\tau))(q) - (\varrho*\nabla\tilde{\psi}(\,.\,,\tau))(\tilde{q})| \leq \|\varrho(\,.\,-q) - \varrho(\,.\,-\tilde{q})\|_{L^2}\|\tilde{\psi}(\,.\,,\tau)\|_{\dot{H}^1}. \quad (110)$$

Now recall that for any two equi-measurable translates Δ_1 and Δ_2 of a bounded domain Δ one has $\|\chi_{\Delta_1} + \chi_{\Delta_2} - \chi_{\Delta_1\cap\Delta_2}\|_{L^2} \leq \sqrt{2|\Delta|}$, where χ_Δ is the characteristic function of Δ. This, the compact support of ϱ, and its Lipschitz continuity, then yield

$$\left|\left(\varrho*\nabla\tilde{\psi}(\,.\,,\tau)\right)(q_\tau) - \left(\varrho*\nabla\tilde{\psi}(\,.\,,\tau)\right)(\tilde{q}_\tau)\right| \leq C_\varrho C_\psi\,|q_\tau - \tilde{q}_\tau|, \quad (111)$$

where $C_\varrho = \sqrt{2|\mathrm{supp}(\varrho)|}\,\mathrm{Lip}\,(\varrho)$, and where we also used $\sup_{\tau\geq 0}\|\tilde{\psi}(\,.\,,\tau)\|_{\dot{H}^1} \leq C_\psi$. We next estimate $|q_\tau - \tilde{q}_\tau|$, for which purpose we (i) use the integrated characteristic equations for q_τ and $\tilde{q}_\tau$, with $q(0) = \tilde{q}(0)$, (ii) pull $|\,.\,|$ under the time integral, (iii) use that $\partial_p\sqrt{1+|p|^2} \in C_b^{0,1}(\mathbb{R}^3)$ with $\mathrm{Lip}\left(\partial_p\sqrt{1+|p|^2}\right)=1$, and obtain

$$|q_\tau - \tilde{q}_\tau| \leq \int_0^\tau\left|\frac{p_{\tau'}}{\sqrt{1+|p_{\tau'}|^2}} - \frac{\tilde{p}_{\tau'}}{\sqrt{1+|\tilde{p}_{\tau'}|^2}}\right|\mathrm{d}\tau'$$
$$\leq \int_0^\tau |p_{\tau'} - \tilde{p}_{\tau'}|\mathrm{d}\tau'. \quad (112)$$

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Repeating steps (i) and (ii) now for $p_{\tau'}$ and $\tilde{p}_{\tau'}$ with $p_0 = \tilde{p}_0$, then applying the triangle inequality to the resulting integrand, followed by applications of (111) and $\sup_{\tau'' \geq 0} \|\psi(\,.\,,\tau'')\|_{\dot{H}^1} \leq C_\psi$, respectively using (107), gives the string of estimates

$$
\begin{aligned}
|p_{\tau'} - \tilde{p}_{\tau'}| &\leq \int_0^{\tau'} \left| \left(\varrho * \nabla\psi(\,.\,,\tau'') \right)(q_{\tau''}) - \left(\varrho * \nabla\tilde{\psi}(\,.\,,\tau'') \right)(\tilde{q}_{\tau''}) \right| \mathrm{d}\tau'' \\
&\leq \int_0^{\tau'} \left| \left(\varrho * \nabla\psi(\,.\,,\tau'') \right)(q_{\tau''}) - \left(\varrho * \nabla\psi(\,.\,,\tau'') \right)(\tilde{q}_{\tau''}) \right| \mathrm{d}\tau'' \\
&\quad + \int_0^{\tau'} \left| \left(\varrho * \nabla\psi(\,.\,,\tau'') - \varrho * \nabla\tilde{\psi}(\,.\,,\tau'') \right)(\tilde{q}_{\tau''}) \right| \mathrm{d}\tau'' \\
&\leq C_\varrho C_\psi \int_0^{\tau'} |q_{\tau''} - \tilde{q}_{\tau''}| \mathrm{d}\tau'' + \|\varrho\|_{L^2} \int_0^{\tau'} \|\psi(\,.\,,\tau'') - \tilde{\psi}(\,.\,,\tau'')\|_{\dot{H}^1} \mathrm{d}\tau'', \qquad (113)
\end{aligned}
$$

with C_ϱ given below (111), and $C_\psi \geq \|\psi_0\|_{\dot{H}^1}$ chosen later. Inserting (113) into (112), recalling from (106) that $\tau \leq t$, then employing a second order variant of the Gronwall lemma (see Appendix A.2) with $z_0 = \tilde{z}_0$, we find for all $\tau \leq t$ that

$$
|q_\tau - \tilde{q}_\tau| \leq \|\varrho\|_{L^2} \int_0^\tau \cosh\left[\underline{w}(\tau - \tau') \right] \int_0^{\tau'} \|\psi(\,.\,,\tau'') - \tilde{\psi}(\,.\,,\tau'')\|_{\dot{H}^1} \mathrm{d}\tau'' \mathrm{d}\tau', \qquad (114)
$$

with $\underline{w} = \sqrt{C_\varrho C_\psi}$. With (114) and (111) we have the relevant estimates for the first term on the right-hand side of (109).

To estimate the second term on r.h.s.(109), we recall that $\partial_p \sqrt{1 + |p|^2} \in C_b^{0,1}(\mathbb{R}^3)$ with $\mathrm{Lip}\left(\partial_p \sqrt{1 + |p|^2} \right) = 1$, then recall (113). Estimating $|q_\tau - \tilde{q}_\tau|$ by r.h.s.(114) and inserting this estimate into (113), we now find that for all $\tau \leq t$ we have

$$
\begin{aligned}
\left| \frac{p_\tau}{\sqrt{1 + |p_\tau|^2}} - \frac{\tilde{p}_\tau}{\sqrt{1 + |\tilde{p}_\tau|^2}} \right| &\leq \|\varrho\|_{L^2} \int_0^\tau \|\psi(\,.\,,\tau') - \tilde{\psi}(\,.\,,\tau')\|_{\dot{H}^1} \mathrm{d}\tau' \\
&\quad + \|\varrho\|_{L^2} \underline{w}^2 \int_0^\tau \int_0^{\tau'} \cosh\left[\underline{w}(\tau' - \tau'') \right] \int_0^{\tau''} \|\psi(\,.\,,\check{\tau}) - \tilde{\psi}(\,.\,,\check{\tau})\|_{\dot{H}^1} \mathrm{d}\check{\tau} \mathrm{d}\tau'' \mathrm{d}\tau', \qquad (115)
\end{aligned}
$$

which provides an upper bound to the second term on r.h.s.(109).

The bounds on the two terms of r.h.s.(109), i.e. (114) with (111), and (115), combine into an estimate of l.h.s.(109) which is independent of z_τ and $\tilde{z}_\tau$, the solutions to the characteristic equations for given fields ψ and $\tilde{\psi}$ with initial data $z_0 = z = \tilde{z}_0$, respectively. We have, independently of z,

$$|\partial_z \mathcal{H}(z_\tau, \tilde\zeta_\tau) - \partial_z \mathcal{H}(\tilde z_\tau, \tilde\zeta_\tau)| \leq \|\varrho\|_{L^2} \int_0^\tau \|\psi(\,.\,, \tau') - \tilde\psi(\,.\,, \tau')\|_{\dot H^1} \mathrm{d}\tau'$$

$$+\|\varrho\|_{L^2}\,\underline{w}^2 \int_0^\tau \cosh\left[\underline{w}(\tau - \tau')\right] \int_0^{\tau'} \|\psi(\,.\,, \tau'') - \tilde\psi(\,.\,, \tau'')\|_{\dot H^1} \mathrm{d}\tau'' \mathrm{d}\tau'$$

$$+\|\varrho\|_{L^2}\,\underline{w}^2 \int_0^\tau \int_0^{\tau'} \cosh\left[\underline{w}(\tau' - \tau'')\right] \int_0^{\tau''} \|\psi(\,.\,, \check\tau) - \tilde\psi(\,.\,, \check\tau)\|_{\dot H^1} \mathrm{d}\check\tau \mathrm{d}\tau'' \mathrm{d}\tau'. \quad (116)$$

We integrate (116) w.r.t. $\mu_0(\mathrm{d}z)$; due to the z-independence of the integrand on the right-hand side that integral factors out there and equals 1. It thus remains to integrate (116) w.r.t. $\mathrm{d}\tau$ from 0 to t, to multiply by e^{-wt} and to take the supremum over $t \geq 0$. The three terms on r.h.s.(116) are estimated by repeating the strategy used in (95) a total of 9 times (however, one of the estimates is just (108) again). For all $w > \underline{w}$ we thereby arrive at the desired estimate for $\sup_{t\geq 0}(e^{-wt}$ second integral on r.h.s.(106)),

$$\sup_{t\geq 0}\left(e^{-wt} \int_0^t \int \left|\partial_z \mathcal{H}(z_\tau, \tilde\zeta_\tau) - \partial_z \mathcal{H}(\tilde z_\tau, \tilde\zeta_\tau)\right| \mathrm{d}\mu_0 \mathrm{d}\tau\right) \leq$$

$$\|\varrho\|_{L^2}\left(\frac{1}{w^2} + \frac{\underline{w}^2}{2w^2(w - \underline{w})} + \frac{\underline{w}^2}{2w^3(w - \underline{w})}\right) \sup_{t\geq 0}\left(e^{-wt}\|\psi(\,.\,, t) - \tilde\psi(\,.\,, t)\|_{\dot H^1}\right), \quad (117)$$

with $\underline{w} = \sqrt{C_\varrho C_\psi}$ from (114). The estimates given by (117) and by (108) (and ensuing text), together with (106), now give the estimate

$$\sup_{t\geq 0}\left(e^{-wt}\|\mu_0 \circ \Pi_{0,t}[\zeta.] - \mu_0 \circ \Pi_{0,t}[\tilde\zeta.]\|_{\mathrm{L}^*}\right) \leq L_2 \sup_{t\geq 0}\left(e^{-wt}\|\psi(\,.\,, t) - \tilde\psi(\,.\,, t)\|_{\dot H^1}\right) \quad (118)$$

whenever $w > \underline{w}$, with

$$L_2[\varrho; w, \underline{w}] = \left(\frac{1}{w} + \frac{1}{w^2} + \frac{(1 + w)\underline{w}^2}{2w^3(w - \underline{w})}\right)\|\varrho\|_{L^2}. \quad (119)$$

Since $\sup_{t\geq 0}\left(e^{-wt}\|\psi(\,.\,, \tau) - \tilde\psi(\,.\,, \tau)\|_{\dot H^1}\right) \leq \sup_{t\geq 0}\left(e^{-wt}\|\zeta_t - \tilde\zeta_t\|_{HL}\right)$, and because of (102), we see that (118) proves (88). Part b) is completed.

We have thus proved that $F_{.,0}(\,.\,|\mathbf{3}_0)$ is a Lipschitz map from a closed subset of $C_{\mathrm{w}}^0(\mathbb{R}_+, \widetilde\Gamma_{\mathrm{B},1}|\mathbf{3}_0)$, defined by the condition $\sup_{t\geq 0}\|\psi(\,.\,, t)\|_{\dot H^1} \leq C_\psi$, to $C_{\mathrm{w}}^0(\mathbb{R}_+, \widetilde\Gamma_{\mathrm{B},1}|\mathbf{3}_0)$ whenever $C_\psi \geq \underline{C}_\psi \geq \|\psi_0\|_{\dot H^1}$ and $w > \underline{w} = \sqrt{C_\varrho C_\psi}$. The Lipschitz constant is

$L[\varrho; w, \underline{w}] = \max\{L_1[\varrho; w], L_2[\varrho; w, \underline{w}]\}$, with L_1 given in (101) and L_2 in (119). Finally, we note that everything proven so far for $F_{.,0}(\,.\,|\mathbf{3}_0)$ remains valid for its truncation $\overline{F}_{.,0}(\,.\,|\mathbf{3}_0)$, obtained for all $t > 0$ by replacing $\Phi^{\psi}_{t,0}[\mu_.]$ by its upper truncation $\overline{\Phi}^{\psi}_{t,0}[\mu_.]$ given in (84). By time reversal symmetry, the same conclusions hold with $C^0_{\rm w}(\mathbb{R}_+, \widetilde{\Gamma}_{\rm B,1}|\mathbf{3}_0)$ replaced by $C^0_{\rm w}(\mathbb{R}, \widetilde{\Gamma}_{\rm B,1}|\mathbf{3}_0)$. This concludes the Lipschitz continuity part of the proof.

We show next that for sufficiently large $\underline{C}_{\psi} \geq \|\psi_0\|_{\dot{H}^1}$ and $\widetilde{R} \geq \|\mathbf{3}^0_.\|_{\rm w}$, the map $F_{.,0}(\,.\,|\mathbf{3}_0)$ sends a closed subset of a closed ball $B_{\widetilde{R}}(\mathbf{3}^0_.) \subset C^0_{\rm w}(\mathbb{R}_+, \widetilde{\Gamma}_{\rm B,1}|\mathbf{3}_0)$, satisfying $\sup_{t\geq 0} \|\psi(\,.\,,t)\|_{\dot{H}^1} \leq C_{\psi}$, into $B_{\widetilde{R}}(\mathbf{3}^0_.)$ whenever $C_{\psi} \geq \underline{C}_{\psi}$ and $w > \underline{w} = \sqrt{C_{\varrho}C_{\psi}}$; recall that $\mathbf{3}^0_. = F_{.,0}(0_.|\mathbf{3}_0) \in C^0_{\rm w}(\mathbb{R}, \widetilde{\Gamma}_{\rm B,1}|\mathbf{3}_0)$ denotes the free evolution of $\mathbf{3}_0(\in \widetilde{\Gamma}_{\rm B,1})$, where $0_.$ is the trivial constant map $(t \mapsto 0) \in C^0_{\rm w}\left(\mathbb{R}, \widetilde{M}_1(\mathbb{R}^6) \oplus \dot{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\right)$. Note that l.h.s.(89) is well-defined if we substitute $0_.$ for $\widetilde{\mathbf{3}}_.$; clearly, $0_. \notin C^0_{\rm w}(\mathbb{R}, \widetilde{\Gamma}_{\rm B,1}|\mathbf{3}_0)$. However, since $\mu_. - 0_. \neq P_1 - P_1$, r.h.s.(89) cannot be directly used to estimate this particular version of l.h.s.(89). One possible way out is to derive an analog of (89) for more general measures involving our extension $\|\,.\,\|_{\widetilde{\rm L}^*}$ of the dual Lipschitz distance, see Appendix A.1. A more direct way out is as follows. We invoke the triangle inequality to estimate $\|\Phi_{t,0}[\mu_.](\zeta_0) - \Phi_{t,0}[0_.](\zeta_0)\|_{HL} \leq \|\Phi_{t,0}[\mu_.](\zeta_0) - \Phi_{t,0}[\mu^0_.](\zeta_0)\|_{HL} + \|\Phi_{t,0}[\mu^0_.](0)\|_{HL}$, where we used that $\Phi_{t,0}[\mu^0_.](\zeta_0) - \Phi_{t,0}[0_.](\zeta_0) = \Phi_{t,0}[\mu^0_.](0)$, and also to estimate

$$\|\mu_0 \circ \Pi_{0,t}[\zeta_.] - \mu_0 \circ \Pi_{0,t}[0_.]\|_{\rm L^*} \leq \|\mu_0 \circ \Pi_{0,t}[\zeta_.] - \mu_0 \circ \Pi_{0,t}[\zeta^0_.]\|_{\rm L^*}$$
$$+ \|\mu_0 \circ \Pi_{0,t}[\zeta^0_.] - \mu_0 \circ \Pi_{0,t}[0_.]\|_{\rm L^*} \qquad (120)$$

Recalling that $F_{.,0}(0_.|\mathbf{3}_0) = \mathbf{3}^0_.$, it is straightforward to verify that (89) modifies to

$$\|F_{.,0}(\mathbf{3}_.|\mathbf{3}_0) - \mathbf{3}^0_.\|_{\rm w} \leq L\|\mathbf{3}_. - \mathbf{3}^0_.\|_{\rm w} + K. \qquad (121)$$

whenever $w > \underline{w} = \sqrt{C_{\varrho}C_{\psi}}$, with $L[\varrho; w, \underline{w}] = \max\{L_1[\varrho; w], L_2[\varrho; w, \underline{w}]\}$ as before, and

$$K = \sup_{t\geq 0}\left(e^{-wt}\left(\|\Phi_{t,0}[\mu^0_.](0)\|_{HL} + \|\mu_0 \circ \Pi_{0,t}[\zeta^0_.] - \mu_0 \circ \Pi_{0,t}[0_.]\|_{\rm L^*}\right)\right). \qquad (122)$$

Since $\|\Phi^{\psi}_{t,0}[\mu^0_.](0)\|_{\dot{H}^1} \leq \|\varrho\|_{L^2}t$ and $\|\Phi^{\varpi}_{t,0}[\mu^0_.](0)\|_{L^2} \leq \|\varrho\|_{L^2}t$ by (33), and since

$$\sup_{t\geq 0}\left(e^{-wt}\|\mu_0 \circ \Pi_{0,t}[\zeta^0_.] - \mu_0 \circ \Pi_{0,t}[0_.]\|_{\rm L^*}\right) \leq L_2 \sup_{t\geq 0}\left(e^{-wt}\|\Phi^{\psi}_{t,0}[\mu^0_.](\zeta_0)\|_{\dot{H}^1}\right) \qquad (123)$$

by (118), with $\|\Phi^{\psi}_{t,0}[\mu^0_.](\zeta_0)\|_{\dot{H}^1} \leq (2\mathcal{E}_{\rm W}(\zeta_0))^{1/2} + \|\varrho\|_{L^2}t$ (by (33) again), we find that

$$K \leq \tfrac{1}{w}\tfrac{2}{e}\|\varrho\|_{L^2} + L_2 \max\left\{\tfrac{1}{w}\|\varrho\|_{L^2}, (2\mathcal{E}_{\rm W}(\zeta_0))^{1/2}\right\}. \qquad (124)$$

Recalling that $L[\varrho; w, \underline{w}] = \max\{L_1[\varrho; w], L_2[\varrho; w, \underline{w}]\}$ with L_1 given in (101) and L_2 in (119), we see that both L and K are monotonically decreasing functions of $w(> \underline{w})$, with asymptotic decay to zero $\propto 1/w$ for large w. Now let w be large enough such that $L \leq 1/2$. Pick an R_*, independent of w, such that $K \leq LR_*$ (clearly such an R_* exists). Now, either $\|\mathbf{3}. - \mathbf{3}^0.\|_{\widetilde{w}} \leq R_*$ or $R_* \leq \|\mathbf{3}. - \mathbf{3}^0.\|_{\widetilde{w}}$. In the former case, $\|F_{.,0}(\mathbf{3}.|\mathbf{3}_0) - \mathbf{3}^0.\|_w \leq 2LR_*$, i.e. $F_{.,0}(\,.\,|\mathbf{3}_0)$ maps any closed subset of the closed ball $B_{R_*}(\mathbf{3}^0.)$ satisfying $\sup_{t\geq 0}\|\psi(\,.\,,t)\|_{\dot{H}^1} \leq C_\psi$ into $B_{2LR_*}(\mathbf{3}^0.)$; clearly, since by assumption w is large enough so that $L \leq 1/2$, the ball $B_{2LR_*}(\mathbf{3}^0.) \subset B_{R_*}(\mathbf{3}^0.)$. In the latter case on the other hand we have $\|F_{.,0}(\mathbf{3}.|\mathbf{3}_0) - \mathbf{3}^0.\|_w \leq 2L\|\mathbf{3}. - \mathbf{3}^0.\|_{\widetilde{w}}$, with $L \leq 1/2$. Thus we conclude that, as long as $\widetilde{R} \geq R_*$, the fixed point map $F_{.,0}(\,.\,|\mathbf{3}_0)$ sends any closed subset of $B_{\widetilde{R}}(\mathbf{3}^0.)$ which satisfies $\sup_{t\geq 0}\|\psi(\,.\,,t)\|_{\dot{H}^1} \leq C_\psi$ into $B_{\widetilde{R}}(\mathbf{3}_0)$, given that $L \leq 1/2$. Again, we note that everything proven in this paragraph for $F_{.,0}(\,.\,|\mathbf{3}_0)$ remains valid for its truncation $\overline{F}_{.,0}(\,.\,|\mathbf{3}_0)$.

It remains to notice that the truncated map $\overline{F}_{.,0}(\,.\,|\mathbf{3}_0)$ sends any closed subset of $C^0_w(\mathbb{R}_+, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$ satisfying $\sup_{t\geq 0}\|\psi(\,.\,,t)\|_{\dot{H}^1} \leq C_\psi$, with $C_\psi \geq \|\psi_0\|_{\dot{H}^1}$, to itself. Hence, for $\widetilde{R} \geq R_*$ and $\underline{C}_\psi \geq \|\psi_0\|_{\dot{H}^1}$, the truncated map $\overline{F}_{.,0}(\,.\,|\mathbf{3}_0)$ sends the intersection of any closed ball $B_{\widetilde{R}}(\mathbf{3}^0.) \subset C^0_w(\mathbb{R}_+, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$ with any closed subsets of $C^0_w(\mathbb{R}_+, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$ satisfying $\sup_{t\geq 0}\|\psi(\,.\,,t)\|_{\dot{H}^1} \leq C_\psi$ to itself whenever $C_\psi \geq \underline{C}_\psi$ and $w > \underline{w} = \sqrt{C_\varrho C_\psi}$ are large enough so that $L \leq 1/2$. For instance, this can be achieved as follows. Setting $w = 2\underline{w}$, the Lipschitz constant becomes

$$L = \max\left\{\sqrt{\frac{\mathrm{Lip}^{\circ 2}(\varrho * \nabla^2 \varrho)}{16\underline{w}^4}} + \sqrt{\frac{\mathrm{Lip}^{\circ 2}(\varrho * \varrho)}{4\underline{w}^2} + \frac{\mathrm{Lip}^{\circ 2}(\varrho * (\Omega_0 \cdot \nabla)^2 \varrho)}{16\underline{w}^4}}\, ,\, \left(1 + \frac{1}{2\underline{w}}\right)\frac{5\|\varrho\|_{L^2}}{8\underline{w}}\right\}, \quad (125)$$

and also setting now $C_\psi = \underline{C}_\psi$ so that $\underline{w} = \sqrt{C_\varrho \underline{C}_\psi}$, we see that there is a unique $\underline{C}^*_\psi[\varrho]$ such that r.h.s.(125)$\leq 1/2$ for $\underline{C}_\psi \geq \underline{C}^*_\psi[\varrho]$; hence, choosing $\underline{C}_\psi \geq \max\{\underline{C}^*_\psi[\varrho]\,,\,\|\psi_0\|_{\dot{H}^1}\}$ will do. This completes the proof of Proposition 3.4.   Q.E.D.

3.3.4   Proof of Theorem 3.5

It suffices in the following to continue to work with $C_\psi = \underline{C}_\psi$ and $w = 2\underline{w}$, as done at the end of the proof of Proposition 3.4. Thus, we need to show that Theorem 3.5 is true for sufficiently large $\underline{C}_\psi \geq \max\{\underline{C}^*_\psi[\varrho]\,,\,\|\psi_0\|_{\dot{H}^1}\}$.

Proof of Theorem 3.5: Let $C_\psi = \underline{C}_\psi \geq \max\{\underline{C}^*_\psi[\varrho]\,,\,\|\psi_0\|_{\dot{H}^1}\}$, and $w = 2\underline{w}$. Then, as shown in the proof of Proposition 3.4, $\overline{F}_{.,0}(\mathbf{3}.|\mathbf{3}_0)$ is a contraction map, with Lipschitz constant $L \leq 1/2$, from the intersection of any closed ball $B_{\widetilde{R}}(\mathbf{3}^0.) \subset C^0_w(\mathbb{R}, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$ of radius $\widetilde{R} \geq R_*$ with any closed subset of $C^0_w(\mathbb{R}, \widetilde{\Gamma}_{B,1}|\mathbf{3}_0)$ defined by the condition

$\sup_{t\geq 0}\|\psi(\,.\,,t)\|_{\dot{H}^1} \leq C_\psi(=\underline{C}_\psi)$, to itself. The standard contraction mapping theorem now guarantees the existence of a unique fixed point $t \mapsto \mathfrak{Z}_t \in C^0_{\mathrm{w}}(\mathbb{R},\widetilde{\Gamma}_{\mathrm{B},1})$, with $t \mapsto \psi(\,.\,,t) \in C^0_b(\mathbb{R},\dot{H}^1(\mathbb{R}^3))$, of the fixed point equation with the truncated F,

$$\mathfrak{Z}_. = \overline{F}_{.,0}(\mathfrak{Z}_.|\mathfrak{Z}_0). \tag{126}$$

We will now show that for sufficiently large $\underline{C}_\psi$ the solutions of (126) are fixed points for F, moreover of type $C^1(\mathbb{R},\widetilde{\Gamma}_{\mathrm{B},1})$, thus furnishing unique $\widetilde{\Gamma}_{\mathrm{B},1}$-strong Vlasov solutions.

To this effect, choose $\underline{C}_\psi > \max\{\underline{C}^*_\psi[\varrho], \sqrt{2\mathcal{E}_{\mathrm{W}}(\zeta_0)}, \sqrt{4 + 4E_0 - 8E_\perp}\}$, where $\mathcal{E}_{\mathrm{W}}(\zeta_0)$ is the initial field energy, $E_0 = \mathcal{E}(\mathfrak{Z}_0)$ is the total energy of the initial state, and where $E_\perp$ is the ground state energy of the N-body Hamiltonian given in (38); note that the ground state energy is N-independent and therefore identical to the ground state of the continuum (Vlasov) limit energy functional (67). Note that automatically we have $\underline{C}_\psi > \|\psi_0\|_{\dot{H}^1}$, for it is trivially obvious that $\|\psi_0\|_{\dot{H}^1} \leq \sqrt{2\mathcal{E}_{\mathrm{W}}(\zeta_0)}$ and easily shown that $\|\psi_0\|_{\dot{H}^1} \leq \sqrt{4 + 4E_0 - 8E_\perp}$. With the so chosen $\underline{C}_\psi$, there then exists at least a small neighborhood of $t = 0$ such that for all t in this neighborhood, the fixed points of (126), which are of type $C^0(\mathbb{R},\widetilde{\Gamma}_{\mathrm{B},1})$, by continuity satisfy

$$\mathfrak{Z}_t = F_{t,0}(\mathfrak{Z}_.|\mathfrak{Z}_0). \tag{127}$$

Now recall that $\psi_0 \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3)$, ensuring a strong solution of the wave equation, by the Hille–Yosida theorem. Next recall the remark after Lemma 2.4; viz.

$$\|\psi(\,.\,,t)\|_{\dot{H}^1} \leq (2\mathcal{E}_{\mathrm{W}}(\zeta_0))^{1/2} + \|\varrho\|_{L^2}|t|, \tag{128}$$

for the strong solution of the wave equation given *any subluminal* source $\varrho * \rho \in C^0(\mathbb{R}, C^\infty_0(\mathbb{R}^3))$. Clearly, there is a unique $\underline{T} > 0$ for which

$$\underline{C}_\psi = (2\mathcal{E}_{\mathrm{W}}(\zeta_0))^{1/2} + \|\varrho\|_{L^2}\underline{T}, \tag{129}$$

such that $\|\psi(\,.\,,t)\|_{\dot{H}^1} < \underline{C}_\psi$ strictly for all $|t| < \underline{T}$, by (128). This now implies that there exist $T \geq \underline{T}$ such that the fixed point $\mathfrak{Z}_.$ of (126) satisfies (127) for all $t \in [-T,T]$. We now show that $\sup\{T : (127) \text{ holds for all } |t| \leq T\} = \infty$.

Thus, suppose that $\sup\{T : (127) \text{ holds for all } |t| \leq T\} = T_* < \infty$. Then for either $t = T^+_*$ or $t = -T^-_*$ (or both), $\mathfrak{Z}_t$ is given by (126) but *not* by (127). For the sake of concreteness, assume that this is so for some t in a right neighborhood of T_*. This then means that for all $t \in (T_*, T_* + \epsilon)$ we have $\|\Phi^\psi_{t,0}\|_{\dot{H}^1} \geq \underline{C}_\psi > \sqrt{4 + 4E - 8E_\perp}$, which in particular implies that $\lim_{t\downarrow T_*} \|\Phi^\psi_{t,0}\|_{\dot{H}^1} \geq \underline{C}_\psi > \sqrt{4 + 4E - 8E_\perp}$. On the other hand,

for all $t \in [-T_*, T_*]$, the solution $\mathfrak{Z}$ of (126) satisfies (127), and ζ is a strong solution of the wave equation. But then, by Corollary 3.7, $t \mapsto \mathfrak{Z}_t \in C^1([-T_*, T_*], \widetilde{\Gamma}_{\mathrm{B},1})$ is a $\widetilde{\Gamma}_{\mathrm{B},1}$-strong solution, which by Theorem 3.3 conserves energy. As a consequence of energy conservation, for all $t \in [-T_*, T_*]$, and in particular for $t = T_*$, we have

$$\|\Phi^{\psi}_{t,0}\|_{\dot{H}^1} \leq \sqrt{4 + 4E - 8E_\perp} \, . \tag{130}$$

Since $t \mapsto \Phi^{\psi}_{t,0} \in C^0(\mathbb{R}, \dot{H}^1(\mathbb{R}^3))$, we thus have a contradiction to the previously concluded $\|\Phi^{\psi}_{t,0}\|_{\dot{H}^1} \geq \underline{C}_\psi > \sqrt{4 + 4E - 8E_\perp}$ for $t > T_*$. Hence, $T_* = \infty$. Q.E.D.

REMARK 3.9 *By the proof of Theorem 3.5 it suffices to work with* $w \geq 2\underline{w}^*$, *where*

$$\underline{w}^* = \sqrt{C_\varrho \max\left\{\underline{C}^*_\psi[\varrho], \sqrt{4 + 4E - 8E_\perp}\right\}}. \tag{131}$$

REMARK 3.10 *In the proof of Theorem 3.5 we only made use of the a priori bound (130) following from Theorem 3.3 and the analog of the proof of our Proposition 2.8 for the regularized Vlasov model. The other bounds expressed in Proposition 2.8 are carried over as follows.*

Let $t \mapsto \mathfrak{Z}(t) \in C^0(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ *be a generalized solution of the wave gravity Vlasov equations which conserves energy* E, *momentum* P, *and angular momentum* J, *and of course mass* $M = 1$. *Then, beside (130), uniformly in* t *we have*

$$\|\varpi(\,.\,,t)\|^2_{L^2} \leq 2E - 2E_\perp \, , \tag{132}$$

$$\int \sqrt{1 + |p|^2} \mu_t(\mathrm{d}z) \leq 1 + E - E_\perp \, , \tag{133}$$

$$6E_\perp - 3E - 3 \leq \int (\varrho * \psi(\,.\,,t))(x)\mu_t(\mathrm{d}z) \leq E - 1 \tag{134}$$

Moreover, (55) and (56) extend to

$$\left|\int p \mu_t(\mathrm{d}z)\right| \leq |P| + \|\psi^{(N)}(\,.\,,t)\|_{\dot{H}^1} \|\varpi^{(N)}(\,.\,,t)\|_{L^2} \, , \tag{135}$$

$$\left|\int p \times x \mu_t(\mathrm{d}z)\right| \leq |J| + (R + |t|) \|\psi^{(N)}(\,.\,,t)\|_{\dot{H}^1} \|\varpi^{(N)}(\,.\,,t)\|_{L^2} \, . \tag{136}$$

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4   The limit $N \to \infty$

We prove first that the $\widetilde{\Gamma}_{\mathrm{B},1}$-strong N-body generalized solutions of the Vlasov model converge $\|\,.\,\|_{\mathrm{w}}$-strongly to solutions when $N \to \infty$. We then specify when these limit solutions are continuum solutions. Finally we discuss the probabilistic import in terms of a law of large numbers and a central limit theorem.

4.1   The $\widetilde{\Gamma}_{\mathrm{B},1}$-strong limit of the N-body generalized solutions

Suppose the family of initial empirical measures converges $\widetilde{\Gamma}_{\mathrm{B},1}$-strongly when $N \to \infty$, written $\varepsilon[\mathbf{z}_0^{(N)}](\mathrm{d}z) \rightsquigarrow \mu_0(\mathrm{d}z)$. Then the microscopic 'density' $\rho^{(N)}(\,.\,,0)$, as given in (18) with $t = 0$, converges strongly (in the marginal measures' subspace) to the 'density' $\rho(\,.\,,0)$. Finally, assume that $\zeta[\mathbf{z}_0^{(N)}] \to \zeta_0 \in \dot{H}^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ satisfying (59), (60) with $\psi_0 \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3)$. Our goal is to show that, when $N \to \infty$, the generalized solution $t \mapsto (\varepsilon[\mathbf{z}_t^{(N)}]; \zeta_t^{(N)}) \in (C_{\mathrm{w}}^0 \cap C^1)(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ associated with this converging family of initial data in turn converges in $\|\,.\,\|_{\mathrm{w}}$ norm to a solution $t \mapsto (\mu_t; \zeta_t) \in (C_{\mathrm{w}}^0 \cap C^1)(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ of the regularized wave gravity Vlasov fixed point equation. In the following, $\mathfrak{Z}_t^{(N)} \xrightarrow{\widetilde{\Gamma}_{\mathrm{B},1}} \mathfrak{Z}_t$ means $\psi^{(N)}(\,.\,,t) \xrightarrow{\dot{H}^1} \psi(\,.\,,t)$ satisfying (59), $\varpi^{(N)}(\,.\,,t) \xrightarrow{L^2} \varpi(\,.\,,t)$ satisfying (60), and $\varepsilon[\mathbf{z}_t^{(N)}] \rightsquigarrow \mu_t$ in $\widetilde{P}_1$.

The main result is an immediate consequence of the following theorem, which states that the $\|\,.\,\|_{\mathrm{w}}$ induced distance between any two $C_{\mathrm{w}}^0(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ solutions of our Vlasov fixed point equation is controlled by the $\widetilde{\Gamma}_{\mathrm{B},1}$ distance of their initial states in $\Gamma_{\mathrm{B},1}$.

PROPOSITION 4.1 *Let* $\mathbb{I} \subset \mathbb{N}$ *or* $\mathbb{I} \subset \mathbb{R}$ *be an index set, and let* $\{\mathfrak{Z}^{(\alpha)} \in C_{\mathrm{w}}^0(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})\}_{\alpha \in \mathbb{I}}$ *be a family of solutions of the Vlasov fixed point equation (76), having initial data* $\mathfrak{Z}_0^{(\alpha)} \in \Gamma_{\mathrm{B},1}$ *with* $\psi_0^{(\alpha)} \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3)$, *for which* $E^* := \sup_{\alpha \in \mathbb{I}} \{\mathcal{E}(\mathfrak{Z}_0^{(\alpha)})\}$ *exists. Define*

$$\bar{w} = \sqrt{C_\varrho \max\left\{ \underline{C}_\psi^*[\varrho] , \sqrt{4 + 4E^* - 8E_\perp} \right\}}, \tag{137}$$

with $C_\varrho = \sqrt{2|\mathrm{supp}(\varrho)|}\,\mathrm{Lip}(\varrho)$ *(cf. text below (111)). Then for all* $w \geq 2\bar{w}$ *there exists a constant* $L_0[\bar{w}]$ *such that for any* $(\alpha, \tilde{\alpha}) \in \mathbb{I}^2$,

$$\|\mathfrak{Z}_{\cdot}^{(\alpha)} - \mathfrak{Z}_{\cdot}^{(\tilde{\alpha})}\|_{\mathrm{w}} \leq L_0 \|\mathfrak{Z}_0^{(\alpha)} - \mathfrak{Z}_0^{(\tilde{\alpha})}\|. \tag{138}$$

Before we prove this proposition, we state and prove its main corollary.

THEOREM 4.2 *Let $t \mapsto \mathfrak{Z}_t^{(N)} \in (C_w^0 \cap C^1)(\mathbb{R}, \widetilde{\Gamma}_{B,1})$ be the $\widetilde{\Gamma}_{B,1}$-strong N-body solution of the regularized wave gravity Vlasov equations (57), (58), (62) with Cauchy data $\mathfrak{Z}_0^{(N)} = \lim_{t \to 0} \mathfrak{Z}_t^{(N)}$ described in Theorem 3.3. Suppose $\mathfrak{Z}_0^{(N)} \xrightarrow{\widetilde{\Gamma}_{B,1}} \mathfrak{Z}_0$, with $\mathfrak{Z}_0$ having mass $M(=1)$, energy E, momentum P, and angular momentum J, and with $\psi_0 \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3)$. Then $\|\mathfrak{Z}_\cdot^{(N)} - \mathfrak{Z}_\cdot\|_w \to 0$, where $t \mapsto \mathfrak{Z}_t \in C_w^0(\mathbb{R}, \widetilde{\Gamma}_{B,1})$ is the unique solution of (76) described in Theorem 3.5. Beside mass, $t \mapsto \mathfrak{Z}_t$ also conserves energy, momentum, and angular momentum. Furthermore, since $\psi_0 \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3)$, we also have $\mathfrak{Z}_\cdot \in C^1(\mathbb{R}, \widetilde{\Gamma}_{B,1})$.*

Proof of Theorem 4.2: By Theorem 3.5, there exist unique type $C_w^0(\mathbb{R}, \widetilde{\Gamma}_{B,1})$ solutions $\mathfrak{Z}_\cdot^{(N)}$, $\mathfrak{Z}_\cdot$ of the fixed point equation (76) for each Cauchy data $\mathfrak{Z}_0^{(N)}$, $\mathfrak{Z}_0 \in \Gamma_{B,1}$, with $\psi_0^{(N)}$, ψ_0 in $(\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3)$, respectively. The latter restriction upgrades the solutions to the wave equation to be strong, which by Lemma 3.6 implies solutions of type $(C_w^0 \cap C^1)(\mathbb{R}, \widetilde{\Gamma}_{B,1})$ which conserve mass, momentum, angular momentum, and energy.

Let $\mathfrak{Z}_\cdot^{(\infty)} \equiv \mathfrak{Z}_\cdot$, and set $\mathbb{I} = \mathbb{N} \cup \{\infty\}$. Since energy is conserved by each solution, $E^* = \sup_{\alpha \in \mathbb{I}} \{\mathcal{E}(\mathfrak{Z}_0^{(\alpha)})\}$ exists. Thus, $\bar{w}$ exists. Pick any $w > 2\bar{w}$. Now Proposition 4.1 applies to our family $\{\mathfrak{Z}_\cdot^{(\alpha)}\}_{\alpha \in \mathbb{I}}$, and since $\|\mathfrak{Z}_0^{(N)} - \mathfrak{Z}_0\| \to 0$ by hypothesis, Proposition 4.1 now implies that $\|\mathfrak{Z}_\cdot^{(N)} - \mathfrak{Z}_\cdot^{(\infty)}\|_w \to 0$. Q.E.D.

To prepare the proof of Proposition 4.1, we will need the following lemmata.

LEMMA 4.3 *Let $\zeta_\cdot \in C_b^0(\mathbb{R}, (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$, with $\sup_{t \geq 0} \|\psi(\,.\,,t)\|_{\dot{H}^1} \leq C_\psi$, and let $\underline{w} = \sqrt{C_\varrho C_\psi}$. Then $\Pi_{t,t'}[\zeta_\cdot] \in C^{0,1}(\mathbb{R}^6, \mathbb{R}^6)$, with Lipschitz constant[11]*

$$\mathrm{Lip}\left(\Pi_{t,t'}[\zeta_\cdot]\right) = \tfrac{1}{\sqrt{2}}(2 + \max\{\underline{w}, 1/\underline{w}\})e^{\underline{w}|t-t'|}. \tag{139}$$

Proof of Lemma 4.3: Let $\psi(\,.\,,t) \in C_b^0(\mathbb{R}, \dot{H}^1(\mathbb{R}^3))$ be given, with $\sup_{t \geq 0} \|\psi(\,.\,,t)\|_{\dot{H}^1} \leq C_\psi$. To unburden notation, let $\psi_t(\,.\,)$ stand for $\psi(\,.\,,t)$. Let $t \mapsto z_t \in \mathbb{R}^6$ and $t \mapsto \tilde{z}_t \in \mathbb{R}^6$ be two distinct solutions of (8), (9) for this $\psi_\cdot$. Proceeding analogously to the steps taken in (112) and (113), this time for $\psi = \tilde{\psi}$, but now allowing $z_0 \neq \tilde{z}_0$, we find

$$|\tilde{q}_t - q_t| \leq |\tilde{q}_{t'} - q_{t'}| + \int_{t'}^t \left| \frac{\tilde{p}_\tau}{\sqrt{1 + |\tilde{p}_\tau|^2}} - \frac{p_\tau}{\sqrt{1 + |p_\tau|^2}} \right| \mathrm{d}\tau$$
$$\leq |\tilde{q}_{t'} - q_{t'}| + \int_{t'}^t |\tilde{p}_\tau - p_\tau| \mathrm{d}\tau, \tag{140}$$

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[11]Incidentally, by Lemma 4.3, the largest Liapunov exponent for $\Pi_{t,t'}[\zeta_\cdot]$ is bounded above by $\underline{w}$.

respectively

$$|\tilde{p}_t - p_t| \leq |\tilde{p}_{t'} - p_{t'}| + \int_{t'}^{t} |\left(\varrho * \nabla\psi_\tau\right)(\tilde{q}_\tau) - \left(\varrho * \nabla\psi_\tau\right)(q_\tau)|\mathrm{d}\tau$$

$$\leq |\tilde{p}_{t'} - p_{t'}| + C_\varrho C_\psi \int_{t'}^{t} |\tilde{q}_\tau - q_\tau|\mathrm{d}\tau \,, \tag{141}$$

with C_ϱ and C_ψ as stated in the lemma. Inserting (141) into (140) and using the second order variant of Gronwall's lemma gives

$$|\tilde{q}_t - q_t| \leq |\tilde{q}_{t'} - q_{t'}| \cosh\left[\underline{w}(t - t')\right] + |\tilde{p}_{t'} - p_{t'}| \tfrac{1}{\underline{w}} \sinh\left[\underline{w}|t - t'|\right], \tag{142}$$

with $\underline{w} = \sqrt{C_\varrho C_\psi}$. Back-substituting (142) into (141) and integrating then gives

$$|\tilde{p}_t - p_t| \leq |\tilde{p}_{t'} - p_{t'}| \cosh\left[\underline{w}(t - t')\right] + |\tilde{q}_{t'} - q_{t'}| \underline{w} \sinh\left[\underline{w}|t - t'|\right]. \tag{143}$$

To get from (142) and (143) to the conclusion of Lemma 4.3, use $\cosh(x) \leq e^{|x|}$ and $\sinh(|x|) \leq e^{|x|}/2$, as well as the familiar $\|\vec{v}\|_2 \leq \|\vec{v}\|_1 \leq \sqrt{2}\|\vec{v}\|_2$ for $\vec{v} \in \mathbb{R}^n$. Q.E.D.

The next lemma transfers control about the flow $\Pi_{t,t'}$ on $\mathbb{R}^6$ to the flow $\Pi_{t,t'}^\dagger$ on $\widetilde{P}_1$.

LEMMA 4.4 *For any symplectomorphism Π on $\mathbb{R}^6$ which in addition is a Lipschitz map with Lipschitz constant Λ, the adjoint map $\Pi^\dagger\colon M(\mathbb{R}^6) \mapsto M(\mathbb{R}^6)$, defined by $\Pi^\dagger(\sigma) := \sigma \circ \Pi^{-1}$, is a positivity- and $\|\,.\,\|_{\mathrm{TV}}$-preserving smooth automorphism of $M(\mathbb{R}^6)$, and it is a Lipschitz homeomorphism on $\widetilde{M}_1(\mathbb{R}^6)$ for $\|\,.\,\|_{\widetilde{L^*}}$ with Lipschitz constant Λ.*

Proof of Lemma 4.4: First, since Π is a symplectomorphism, by way of the definition of its adjoint, $\Pi^\dagger$ maps $M(\mathbb{R}^6)$ smoothly onto $M(\mathbb{R}^6)$, and it preserves (a) $\|\sigma\|_{\mathrm{TV}}$ for $\sigma \in M$ and (b) the positivity of $\mu \in M_+$. Furthermore, since Π is invertible, so is $\Pi^\dagger$.

To see that $\Pi^\dagger$ is a homeomorphism of $\widetilde{M}_1(\mathbb{R}^6)$, we only need to show that $\Pi^\dagger$ maps $\widetilde{M}_1$ into $\widetilde{M}_1$. Thus, let $z_* \in \mathbb{R}^6$ be the unique element of $\ker\Pi$. Then note that by the definition of $\Pi^\dagger$ and the Lipschitz property of Π we have

$$\left|\int |z|\sigma \circ \Pi^{-1}(\mathrm{d}z)\right| = \left|\int |\Pi(z) - \Pi(z_*)|\sigma(\mathrm{d}z)\right| \leq \Lambda \int |z - z_*||\sigma|(\mathrm{d}z)\,, \tag{144}$$

where $|\sigma|$ is the total variation of σ; the last integral exists for $\sigma \in \widetilde{M}_1$.

As for the Lipschitz continuity of the adjoint flow, let $\hat{\sigma}, \check{\sigma} \in \widetilde{M}_1(\mathbb{R}^6)$. We have

$$\|\Pi^\dagger(\hat\sigma) - \Pi^\dagger(\check\sigma)\|_{\mathrm{L}^*} = \sup_{g\in C^{0,1}(\mathbb{R}^6)} \left\{ \left|\int g\,\mathrm{d}\big(\hat\sigma\circ\Pi^{-1} - \check\sigma\circ\Pi^{-1}\big)\right| : \mathrm{Lip}\,(g)\le 1 \right\}$$

$$= \sup_{g\in C^{0,1}(\mathbb{R}^6)} \left\{ \left|\int g\circ\Pi\,\mathrm{d}(\hat\sigma - \check\sigma)\right| : \mathrm{Lip}\,(g)\le 1 \right\}$$

$$= \Lambda \sup_{g\in C^{0,1}(\mathbb{R}^6)} \left\{ \left|\int \frac{1}{\Lambda} g\circ\Pi\,\mathrm{d}(\hat\sigma - \check\sigma)\right| : \mathrm{Lip}\,(g)\le 1 \right\}$$

$$\le \Lambda\|\hat\sigma - \check\sigma\|_{\mathrm{L}^*}. \tag{145}$$

In the last step, we used that $\Lambda^{-1}g\circ\Pi \in C^{0,1}(\mathbb{R}^6)$ with $\mathrm{Lip}\,(\Lambda^{-1}g\circ\Pi)\le 1$.   Q.E.D.

Proof of Proposition 4.1: Pick $w > 2\bar{w}$, with $\bar{w}$ defined in (137), and pick $\mathfrak{Z}_{\cdot}, \widetilde{\mathfrak{Z}}_{\cdot} \in C^0_{\mathrm{w}}(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ from the family of solutions $\mathfrak{Z}^{(\alpha)}_{\cdot}$ of the Vlasov fixed point equation (76) specified in Proposition 4.1. Then

$$\|\mathfrak{Z}_{\cdot} - \widetilde{\mathfrak{Z}}_{\cdot}\|_{\mathrm{w}} = \|F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\mathfrak{Z}_0) - F_{\cdot,0}(\widetilde{\mathfrak{Z}}_{\cdot}|\widetilde{\mathfrak{Z}}_0)\|_{\mathrm{w}}. \tag{146}$$

By the triangle inequality,

$$\|F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\mathfrak{Z}_0) - F_{\cdot,0}(\widetilde{\mathfrak{Z}}_{\cdot}|\widetilde{\mathfrak{Z}}_0)\|_{\mathrm{w}} \le \|F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\mathfrak{Z}_0) - F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\widetilde{\mathfrak{Z}}_0)\|_{\mathrm{w}}$$
$$+ \|F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\widetilde{\mathfrak{Z}}_0) - F_{\cdot,0}(\widetilde{\mathfrak{Z}}_{\cdot}|\widetilde{\mathfrak{Z}}_0)\|_{\mathrm{w}}. \tag{147}$$

Now, $\|F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\widetilde{\mathfrak{Z}}_0) - F_{\cdot,0}(\widetilde{\mathfrak{Z}}_{\cdot}|\widetilde{\mathfrak{Z}}_0)\|_{\mathrm{w}}$ was estimated already in the proof of Proposition 3.4, see (89) (recall that the conditioning $\lim_{t\to 0}\mathfrak{Z}_t = \mathfrak{Z}_0 = \lim_{t\to 0}\widetilde{\mathfrak{Z}}_t$ that entered the statement of Proposition 3.4 did not enter the estimates for (89) themselves). Furthermore, with $w > 2\bar{w}$ it follows that the parameter conditions in the proof of Theorem 3.5 are fulfilled for each $\widetilde{\mathfrak{Z}}_0$; hence, in (89) we have $L[\varrho; w, \underline{w}] \le 1/2$ for each $\widetilde{\mathfrak{Z}}_0$. Thus, by (146), (147), and (89), and with $(1 - L[\varrho; w, \underline{w}])^{-1} \le 2$, we arrive at the estimate

$$\|\mathfrak{Z}_{\cdot} - \widetilde{\mathfrak{Z}}_{\cdot}\|_{\mathrm{w}} \le 2\|F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\mathfrak{Z}_0) - F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\widetilde{\mathfrak{Z}}_0)\|_{\mathrm{w}}, \tag{148}$$

uniformly for all $\widetilde{\mathfrak{Z}}_0 \in \{\mathfrak{Z}^{(\alpha)}_0\}_{\alpha\in\mathbb{I}}$.

The proof of Proposition 4.1 has thus been reduced to proving Lipschitz continuity of $F_{\cdot,0}$ in its second argument, given the first. Since, by the triangle inequality,

$$\|F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\mu_0; \zeta_0) - F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\tilde\mu_0; \tilde\zeta_0)\|_{\mathrm{w}} \le \|F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\mu_0; \zeta_0) - F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\tilde\mu_0; \zeta_0)\|_{\mathrm{w}}$$
$$+ \|F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\tilde\mu_0; \zeta_0) - F_{\cdot,0}(\mathfrak{Z}_{\cdot}|\tilde\mu_0; \tilde\zeta_0)\|_{\mathrm{w}}, \tag{149}$$

it suffices to show that for given $\mathbf{3}$ and ζ_0, we have

$$\|F_{.,0}(\mathbf{3}.|\mu_0;\zeta_0) - F_{.,0}(\mathbf{3}.|\tilde{\mu}_0;\zeta_0)\|_{\mathrm{w}} \leq L_1^* \|\mu_0 - \tilde{\mu}_0\|_{\mathrm{L}^*}, \tag{150}$$

and for given $\mathbf{3}$ and $\tilde{\mu}_0$,

$$\|F_{.,0}(\mathbf{3}.|\tilde{\mu}_0;\zeta_0) - F_{.,0}(\mathbf{3}.|\tilde{\mu}_0;\tilde{\zeta}_0)\|_{\mathrm{w}} \leq L_2^* \|\zeta_0 - \tilde{\zeta}_0\|_{HL}, \tag{151}$$

with L_1^*, L_2^* depending at most on $\bar{w}$. For then it follows from (149), (150), (151) that

$$\|F_{.,0}(\mathbf{3}.|\mathbf{3}_0) - F_{.,0}(\mathbf{3}.|\tilde{\mathbf{3}}_0)\|_{\mathrm{w}} \leq L^* \|\mathbf{3}_0 - \tilde{\mathbf{3}}_0\|, \tag{152}$$

with $L^*[\bar{w}] := \max\{L_1^*, L_2^*\}$, completing the proof of Proposition 4.1, with $L_0 = 2L^*$.

As to (150), for all $t \in \mathbb{R}$ we have

$$\|F_{t,0}(\mathbf{3}.|\mu_0;\zeta_0) - F_{t,0}(\mathbf{3}.|\tilde{\mu}_0;\zeta_0)\| = \|\Pi_{t,0}^\dagger[\zeta.](\mu_0) - \Pi_{t,0}^\dagger[\zeta.](\tilde{\mu}_0)\|_{\mathrm{L}^*}$$
$$\leq \tfrac{2+\max\{\bar{w},1/\bar{w}\}}{\sqrt{2}} e^{\bar{w}|t|} \|\mu_0 - \tilde{\mu}_0\|_{\mathrm{L}^*}, \tag{153}$$

the inequality by Lemma 4.3 and Lemma 4.4. Since $w \geq 2\bar{w}$, the $\sup_{t \in \mathbb{R}}\left(e^{-w|t|}(153)\right)$ exists. Estimating it further with $w - \bar{w} \geq \bar{w}$ for $w \geq 2\bar{w}$ now gives (150), with

$$L_1^*[\bar{w}] = \frac{2 + \max\{\bar{w}, 1/\bar{w}\}}{\sqrt{2}}. \tag{154}$$

As to (151), for all $t \in \mathbb{R}$ we have

$$\|F_{t,0}(\mathbf{3}.|\tilde{\mu}_0;\zeta_0) - F_{t,0}(\mathbf{3}.|\tilde{\mu}_0;\tilde{\zeta}_0)\| = \|\Phi_{t,0}^\psi[0.](\zeta_0 - \tilde{\zeta}_0)\|_{\dot{H}^1} + \|\Phi_{t,0}^\varpi[0.](\zeta_0 - \tilde{\zeta}_0)\|_{L^2}, \tag{155}$$

where $\Phi_{t,0}^\psi[0.](.)$ and $\Phi_{t,0}^\varpi[0.](.)$ are the free propagators obtained from Kirchhoff's formulas (81) and (82) by replacing $\mu_. \to 0.$; note that $\Phi_{t,0}^\psi[0.](.)$ and $\Phi_{t,0}^\varpi[0.](.)$ are linear operators. For initial data $\psi_0 \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3)$, the freely propagating wave is a HL-strong solution of the homogeneous wave equation and its field energy $\mathcal{E}_{\mathrm{W}}(\zeta_.^{\mathrm{free}})$ is conserved. This implies the bounds

$$\|\Phi_{t,0}^\psi[0.](\zeta_0 - \tilde{\zeta}_0)\|_{\dot{H}^1} + \|\Phi_{t,0}^\varpi[0.](\zeta_0 - \tilde{\zeta}_0)\|_{L^2} \leq 2\sqrt{\mathcal{E}_{\mathrm{W}}(\zeta_0 - \tilde{\zeta}_0)} \leq \sqrt{2}\|\zeta_0 - \tilde{\zeta}_0\|_{HL}. \tag{156}$$

Hence,

$$L_2^* = \sqrt{2}. \tag{157}$$

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Estimates (150) with (153), and (151) with (157) now combine to

$$\|F_{.,0}(\mathbf{3}.|\mu_0;\zeta_0) - F_{.,0}(\mathbf{3}.|\tilde{\mu}_0;\tilde{\zeta}_0)\|_{\mathrm{w}} \le L^*\|\mathbf{3}_0 - \widetilde{\mathbf{3}}_0\| \tag{158}$$

with

$$L^*[\bar{w}] = \max\left\{\sqrt{2}\,,\, \tfrac{2+\max\{\bar{w},1/\bar{w}\}}{\sqrt{2}}\right\} \tag{159}$$

for all $w > 2\bar{w}$.

The proof of Proposition 4.1 is complete, with $L_0[\bar{w}] = 2L^*[\bar{w}]$.   Q.E.D.

4.2   The continuum limit

Note that so far nothing prevents the measure $\mu_0(\mathrm{d}z)$, which obtains as limit of the $\varepsilon[\mathbf{z}_0^{(N)}](\mathrm{d}z)$ when $N \to \infty$, from being as singular as the $\varepsilon[\mathbf{z}_0^{(N)}](\mathrm{d}z)$ are. In particular, we may even allow $\varepsilon[\mathbf{z}_0^{(N)}](\mathrm{d}z) \rightsquigarrow \delta_{z_0}(\mathrm{d}z)$. Since in physical applications of Vlasov theory one is typically interested in continuum solutions, we now suppose that when $N \to \infty$, the familiy of initial empirical measures $\varepsilon[\mathbf{z}_0^{(N)}](\mathrm{d}z)$ converges $\widetilde{\Gamma}_{\mathrm{B},1}$-strongly to a measure $\mu_0(\mathrm{d}z)$ which is absolutely continuous w.r.t. Lebesgue measure. We write $\mu(\mathrm{d}z) = \mu^f(\mathrm{d}z) = f(z)\mathrm{d}z$ for the absolutely continuous measures in $P_1(\mathbb{R}^6)$. The set of their Radon–Nikodym derivatives is denoted $L^{1,1}_{+,1}(\mathbb{R}^6)$; thus $f \in L^{1,1}_{+,1}(\mathbb{R}^6)$. We now show that when $\mu_0(\mathrm{d}z) = \mu^{f_0}(\mathrm{d}z)$, then $\mu_t = \mu^f_t$, with $f(.\,,.\,,t) \in L^{1,1}_{+,1}(\mathbb{R}^6)$ for all $t \in \mathbb{R}$.

PROPOSITION 4.5 *If* $(\mu_.,\zeta_.) \in (C^0_{\mathrm{w}} \cap C^1)(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ *solves the Vlasov fixed point equation (76) with* $\mu_0 = \mu^{f_0}$, $f_0 \in (L^{1,1}_{+,1} \cap L^p)(\mathbb{R}^6)$ *for some* $p \ge 1$, *then* $\mu_. = \mu^{f(.,.,t)}$ *with* $f(.\,,.\,,t) \in (L^{1,1}_{+,1} \cap L^p)(\mathbb{R}^6)$ *for all* $t \in \mathbb{R}$; *note that* $p \ge 1$ *includes the case that* $f_0 \in L^{1,1}_{+,1}(\mathbb{R}^6)$ *while* $f_0 \notin L^p(\mathbb{R}^6)$ *for any* $p > 1$.

Proof of Proposition 4.5: Suppose $\mu_. \in C^1(\mathbb{R}, \widetilde{P}_1)$ is a strong generalized solution of the Vlasov continuity equation (62) for given $\zeta_. \in (C^0_b \cap C^1)(\mathbb{R}, (\dot{H}^1 \oplus L^2)(\mathbb{R}^3))$, with Cauchy data $\mu_0 = \mu^{f_0}$, $f_0 \in (L^{1,1}_{+,1} \cap L^p)(\mathbb{R}^6)$ for some $p \ge 1$. Then $\mu_. = \mu^{f(.,.,t)}$ with $f(.\,,.\,,t) \in (L^{1,1}_{+,1} \cap L^p)(\mathbb{R}^6)$ for all $t \in \mathbb{R}$. But this follows from the definition of a generalized solution, a straightforward change of variables from z to $\Pi_{t,0}[\zeta_.](z)$ under the integral, noting the properties of the flow $\Pi_{.,.}$ summarized in Corollary 3.7.   Q.E.D.

4.2.1 Additional conservation laws for continuum solutions

The argument used to prove Proposition 4.5 has the useful corollary that continuum solutions $\mathfrak{Z}^f_\cdot$ with $f_0 \in (L^{1,1}_{+,1} \cap L^p)(\mathbb{R}^6)$ for some $p > 1$ enjoy additional conservation laws. Here we wrote $\mathfrak{Z}^f_\cdot$ for $\mathfrak{Z}_\cdot = (\mu_\cdot, \zeta_\cdot)$ with $\mu_t = \mu^{f(\cdot,\cdot,t)}$.

For any $g : \mathbb{R}_+ \to \mathbb{R}$, we define the g-Casimir functional of $\mathfrak{Z}^f$ by

$$\mathcal{C}^{(g)}\left(\mathfrak{Z}^f\right) = \int g \circ f \, \mathrm{d}z \,, \qquad \text{whenever} \quad g \circ f \in L^1(\mathbb{R}^6) \,. \tag{160}$$

For $g = \mathrm{id}$, we obtain the mass functional (64) for absolutely continuous $\mu_t = \mu^f_t$; for $g(\,.\,) = (\mathrm{id}(\,.\,))^p$, $p > 1$, we get the p-th power of the L^p norm of f; the case $g(\,.\,) = -\mathrm{id}(\,.\,)\log(\mathrm{id}(\,.\,)/f_*)$, gives the entropy of f relative to f_*,

$$\mathcal{C}^{(-\mathrm{id}\log(\mathrm{id}/f_*))}\left(\mathfrak{Z}^f\right) = -\int f \ln(f/f_*)\mathrm{d}z \equiv \mathcal{S}(f|f_*); \tag{161}$$

here, $f_* \in L^{1,1}_{+,1}(\mathbb{R}^6)$ is an otherwise arbitrary probability density function. In particular, $\mathcal{S}(f|f_*)$ is well-defined if $f \in (L^{1,1}_{+,1} \cap L^{1+\epsilon})(\mathbb{R}^6)$ for some $\epsilon > 0$.

PROPOSITION 4.6 *Let $t \mapsto \mathfrak{Z}_t \in (C^0_{\mathrm{w}} \cap C^1)(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ be a generalized solution of the regularized wave gravity Vlasov model for which $\mu_t = \mu^f_t$ is absolutely continuous. Then, beside the conservation laws (73), (74), (75), whenever $\mathcal{C}^{(g)}(\mathfrak{Z}^f_0)$ exists, also*

$$\mathcal{C}^{(g)}\left(\mathfrak{Z}^f_\cdot\right) = \mathcal{C}^{(g)}(\mathfrak{Z}^f_0)\,. \tag{162}$$

In particular, if $f_0 \in (L^{1,1}_{+,1} \cap L^{1+\epsilon})(\mathbb{R}^6)$ for some $\epsilon > 0$, then the relative entropy of $f(\,.\,,\,.\,,t) \in L^{1,1}_{+,1}(\mathbb{R}^6)$ is conserved, i.e.

$$\mathcal{S}(f|f_*) = \mathcal{S}(f_0|f_*)\,. \tag{163}$$

4.3   Law of large numbers and central limit theorem

4.3.1   The law of large numbers

Convergence in KR topology of probability measures $\mu^{(N)}$ as $N \to \infty$ implies the convergence in probability of a family of random variables with laws $\mu^{(N)}$; see [Dud02]. Since at time t the *empirical* measures $\varepsilon[\mathbf{z}^{(N)}_t]$ do converge in KR topology to μ_t if they do so at $t = 0$, our theorem about the $N \to \infty$ limit of the N-body generalized solutions to the regularized wave gravity Vlasov model is equivalent to the following law of large numbers.

THEOREM 4.7 *For $N \in \mathbb{N}$, let $\mu_0^{(N)} \in P_1(\mathbb{R}^6)$ be given, with $\operatorname{supp}(\mu_0(\mathrm{d}x \times \mathbb{R}^3)) \subset B_R$, and suppose $\mu_0^{(N)} \rightsquigarrow \mu_0 \in P_1(\mathbb{R}^6)$. Moreover, let $\mathbf{z}_0^{(N)} \in \mathbb{R}^{6N}$ be a random vector whose components in $\mathbb{R}^6$ are (not necessarily independent) random variables $z_1^{(N)}(0), ..., z_N^{(N)}(0)$ with common law $\mu_0^{(N)}$. To each $\mathbf{z}_0^{(N)}$ assign a unique $\zeta[\mathbf{z}_0^{(N)}] \in ((\dot{H}^1 \cap \dot{H}^2) \oplus L^2)(\mathbb{R}^3)$ satisfying (59),(60), such that $\zeta[\mathbf{z}_0^{(N)}] \to \zeta_0 = (\psi(\,.\,,0), \varpi(\,.\,,0)) \in ((\dot{H}^1 \cap \dot{H}^2) \oplus L^2)(\mathbb{R}^3)$ HL-strongly when $N \to \infty$, with $\psi(\,.\,,0), \varpi(\,.\,,0)$ satisfying (59), (60).*

Let $(\mu_., \zeta_.) \in (C_\mathrm{w}^0 \cap C^1)(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ be the unique generalized strong solution of the regularized wave gravity Vlasov equations for initial data (μ_0, ζ_0). Let $t \mapsto \mathbf{z}_t^{(N)} \in \mathbb{R}^{6N}$ be the path in particle phase space whose N components in $\mathbb{R}^6$ jointly solve the Einstein–Newton equations of motion (19), (20) with the initial data $\mathbf{z}_0^{(N)}$, and with $t \mapsto \psi^{(N)}(\,.\,,t)$ and $t \mapsto \varpi^{(N)}(\,.\,,t)$ solving the regularized wave gravity equations (16) and (17) with initial data $\zeta[\mathbf{z}_0^{(N)}]$. Then, for any $g \in C^{0,1}(\mathbb{R}^6)$ and for all t, in probability we have

$$\frac{1}{N}\sum_{i=1}^{N} g\big(z_i^{(N)}(t)\big) \stackrel{N \to \infty}{\longrightarrow} \int g(z)\mu_t(\mathrm{d}z). \qquad (164)$$

REMARK 4.8 *By invoking the extremal decomposition of permutation invariant probability measures on $(\mathbb{R}^6)^{\mathbb{N}}$ [HeSa55], our LLN (164) can be generalized to*

$$\binom{N}{n}^{-1} \sum_{1 \le i_1 < ... < i_n \le N} g\big(z_{i_1}^{(N)}(t), ..., z_{i_n}^{(N)}(t)\big) \stackrel{N \to \infty}{\longrightarrow} \int g(z_1, ..., z_n)\mu_t^{\times n}(\mathrm{d}z_1...\mathrm{d}z_n) \qquad (165)$$

for any permutation invariant $g \in C^{0,1}(\mathbb{R}^{6n})$ and for all t.

REMARK 4.9 *One actually should also allow the field initial data $\zeta_0^{(N)}$ to be random variables independently of the particle random variables for each N, but this would require a whole new setup involving probability measures on field space, the choice of an adequate topology on that space, beyond what has been developed in this paper.*

Even though our LLN does not demand that $\mu_.$ be a continuum solution, in applications this is typically so. While our LLN implies that the Vlasov continuum approximation to the sampling of N body systems becomes exact for all t in the limit of infinite N, for any particular physical system N is fixed and may vary only from system to system. Thus, take $\mu_0^{(N)} = \mu_0$ for all N, with $\mu_0 = \mu^{f_0}$. By assumption $\varepsilon[\mathbf{z}_0^{(N)}] \rightsquigarrow \mu_0$ when $N \to \infty$. Yet for any finite N we have $\|\varepsilon[\mathbf{z}_0^{(N)}] - \mu_0\|_{\mathrm{L}^*} > 0$, and then our estimates of section 4.1 show that at any other time t we only have $\|\varepsilon[\mathbf{z}_t^{(N)}] - \mu_t\|_{\mathrm{L}^*} \le e^{C|t|}\|\varepsilon[\mathbf{z}_0^{(N)}] - \mu_0\|_{\mathrm{L}^*}$. Hence, we can only conclude that the physical mean values l.h.s.(164) at time t can be computed in acceptable approximation by their Vlasov continuum analog, i.e. r.h.s.(164) with $\mu_t = \mu^{f(\cdot,\cdot,t)}$, if $|t|$ is "not too large," a notion which depends on N and on how good the approximation is initially.

4.3.2 The central limit theorem

Having obtained the law of large numbers, we next inquire into the fluctuations around the deterministic mean. Our goal is to derive a central limit theorem for the dynamical variables $(\mathbf{z}_t^{(N)}, \zeta_t^{(N)})$, which are random variables through their dependence on the random initial data for the N particles, viz. $\mathbf{z}_0^{(N)}$.

We adapt the technique of [BrHe77], who studied fluctuations of particle motions for non-relativistic Vlasov equations. This is done in two steps. First we study the differences of (primarily) test particle motions generated by the finite N versus the continuum flows, and of similar type differences of field evolutions. The attribute "primarily" in parentheses refers to the fact that almost all (w.r.t. Lebesgue measure) initial conditions launch a test particle evolution, with finitely many exceptions which are upgraded and follow the proper evolution. In our case all these particle evolutions are generated by the adjoint flows on particle phase space of the Vlasov flows on $\widetilde{P}_1$ whose fixed points are the proper Vlasov evolutions (both of course coupled to the same wave gravity equations). We prove the convergence of the characteristic function of the fluctuation process to a Gaussian characteristic function in a suitable norm. In the second step we extract from this analysis the fluctuations for the proper evolutions.

We recall that initial data $\mathbf{z}_0^{(N)} = (z_1^{(N)}(0), ..., z_N^{(N)}(0)) \in \mathbb{R}^{6N}$ with $q_k^{(N)}(0) \in B_R$ for $k = 1, 2, ..., N$ uniquely define initial data $(\varepsilon[\mathbf{z}_0^{(N)}], \zeta[\mathbf{z}_0^{(N)}]) \in \Gamma_{B,1}$ which launch the unique strong solution $(\varepsilon[\mathbf{z}_{\cdot}^{(N)}], \zeta_{\cdot}^{(N)}) \in (C_w^0 \cap C^1)(\mathbb{R}, \widetilde{\Gamma}_{B,1})$ of our regularized wave gravity Vlasov equations. By (79), (80), the solution $(\varepsilon[\mathbf{z}_{\cdot}^{(N)}], \zeta_{\cdot}^{(N)})$ generates a single particle flow $\Pi_{\cdot,\cdot}[\zeta_{\cdot}^{(N)}](.)$ on $\mathbb{R}^6$ giving single particle evolutions

$$z_t(z_0; \mathbf{z}_0^{(N)}) = \Pi_{t,0}[\zeta_{\cdot}^{(N)}](z_0)\,, \tag{166}$$

and a flow $\Phi_{\cdot,\cdot}[\varepsilon[\mathbf{z}_{\cdot}^{(N)}]](.)$ on field space giving field evolutions

$$\zeta_t(\zeta_0; \mathbf{z}_0^{(N)}) = \Phi_{t,0}[\varepsilon[\mathbf{z}_{\cdot}^{(N)}]](\zeta_0)\,, \tag{167}$$

with data $z_0 \in \mathbb{R}^6$ and $\zeta_0 \in (\dot{H}^1 \oplus L^2)(\mathbb{R}^3)$; note, however, that we are *exclusively* considering data $(z_0, \zeta_0) \in \Gamma_B^{(1)}$. As for the notation, by the r.h.s. of (166) the dynamics $z_{\cdot}$ depends on z_0 and on $\zeta_{\cdot}^{(N)}$, but the latter in turn is implicitly fixed by $\mathbf{z}_0^{(N)}$ (and our Vlasov equations); similarly the notation in (167) is explained. As for their dynamical significance, the dynamics $z_{\cdot}$ solves the *characteristic equations* of the Vlasov continuity equation given the fields $\zeta_{\cdot}^{(N)}$. For almost all data $z_0 \in \mathbb{R}^6$ this is a *test particle* dynamics, the exception being when $z_0 \in \{z_1^{(N)}(0), ..., z_N^{(N)}(0)\}$, in which case $z_{\cdot}$ coincides with one of the components $z_k^{(N)}(\,.\,)$ of the unique solution $(\mathbf{z}_{\cdot}^{(N)}, \zeta_{\cdot}^{(N)})$ of equations (16),

(17), (18), (19), (20) with Cauchy data $(\mathbf{z}_0^{(N)}, \zeta[\mathbf{z}_0^{(N)}])$. The wave dynamics ζ in turn solves the linear inhomogeneous wave equation given the source term $\varrho * \rho^{(N)}$ obtained from $\varepsilon[\mathbf{z}^{(N)}]$ Note that the dynamics of z and ζ are in general independent of each other; the exception occurs when $\zeta_0 = \zeta[\mathbf{z}_0^{(N)}]$, in which case $\zeta = \zeta^{(N)}$.

When $N \to \infty$ such that $\varepsilon[\mathbf{z}_t^{(N)}] \rightsquigarrow \varepsilon[\mathbf{z}_t^{(\infty)}]$ and $\zeta_t^{(N)} \to \zeta_t^{(\infty)}$ (HL-strongly) for all t, the flows for (166), (167) converge to the corresponding flows generated by the Vlasov solution $t \mapsto (\varepsilon[\mathbf{z}_t^{(\infty)}], \zeta_t^{(\infty)})$; note that flows analogous to those for (166), (167) are defined for any admissible solution $\in (C_{\mathrm{w}}^0 \cap C^1)(\mathbb{R}, \widetilde{\Gamma}_{\mathrm{B},1})$ of our Vlasov equations. The discussion of the finite-N fluctuations amounts to analyzing the difference of the evolutions (166), (167) for finite N versus $N = \infty$. Note that in either case the Vlasov dynamics that generates (166) and (167) is determined by $\mathbf{z}_0^{(N)}$, respectively $\mathbf{z}_0^{(\infty)}$.

As to our notation, above we use the symbol $\zeta^{(\infty)}$ to distinguish the limit fields of $\zeta^{(N)}$ from the fields ζ solving (167) in general. However, eventually we will choose $\zeta_0 \equiv \zeta_0^{(\infty)}$ for the sake of concreteness in the CLT. Note furthermore that $\mathbf{z}_0^{(\infty)}$ could be anything from a single point z_* to a continuous function, i.e. we may have $\mathbf{z}_0^{(\infty)} = f_0(z)$ (in distribution). Therefore, $\varepsilon[\mathbf{z}_0^{(\infty)}]$ could be any probability measure $\mu_0 \in P_1$ with $\mathrm{supp}\,(\mu_0(\mathrm{d}x \times \mathbb{R}^3)) \subset B_R$; in particular, if $\mathbf{z}_0^{(\infty)} = f_0(z)$ is an empirical continuum density, the empirical continuum measure $\mu^{f_0}(\mathrm{d}z) = \varepsilon[\mathbf{z}_0^{(\infty)}](\mathrm{d}z)$.

We stipulate further notation. Recall that $(z, \zeta) = \mathfrak{z} \in \Gamma_{\mathrm{B}}^{(1)}$ denotes generic points in $\Gamma_{\mathrm{B}}^{(1)}$. In this vein, for solutions of (166), (167) we use whenever possible the shorthand

$$(z_t, \zeta_t) = \mathfrak{z}_t \tag{168}$$

but when necessary to discuss the components in more detail, we write

$$(z_t, \zeta_t) = (q_t, p_t, \psi_t, \varpi_t). \tag{169}$$

Note that $\mathfrak{z}_t$ is a function of $\mathfrak{z}_0$ and $\mathbf{z}_0^{(N)}$; in components, q_t and p_t are points in $\mathbb{R}^3$ which are functions of z_0 and $\mathbf{z}_0^{(N)}$ while ψ_t and ϖ_t are points in $\dot{H}^1 \cap \dot{H}^2$, respectively L^2, which are functions of ζ_0 and $\mathbf{z}_0^{(N)}$. We also recycle some of our previously stipulated abbreviations. Thus, $\int ..\mu(\mathrm{d}z)$ stands for $\iint_{\mathbb{R}^6} ..\mu(\mathrm{d}x\mathrm{d}p)$ and $\iint ..\mu(\mathrm{d}z)\nu(\mathrm{d}z')$ for $\iint_{\mathbb{R}^6 \times \mathbb{R}^6} ..\mu(\mathrm{d}x\mathrm{d}p)\nu(\mathrm{d}x'\mathrm{d}p')$. So much for notation.

As a technical primer which will allow us to perform estimates needed for the main theorem, we will first show that $\mathfrak{z}_t(\mathfrak{z}_0, \mathbf{z}_0^{(N)})$ is regular as a function of $\mathbf{z}_0^{(N)}$, with bounded derivatives. Since $\mathbf{z}_0^{(N)}$ is uniquely identified with the empirical measure $\varepsilon[\mathbf{z}_0^{(N)}]$ and similarly $\mathbf{z}_0^{(\infty)}$ is uniquely identified with the empirical measure $\varepsilon[\mathbf{z}_0^{(\infty)}]$ (with the

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possibility $\varepsilon[\mathbf{z}_0^{(\infty)}](\mathrm{d}z) = \mu^{f_0}(\mathrm{d}z)$, the continuum case), we really mean regularity of $\mathfrak{z}_t$ as a function of $\varepsilon[\mathbf{z}_0^{(N)}]$, respectively $\varepsilon[\mathbf{z}_0^{(\infty)}]$, with bounded derivatives w.r.t. $\varepsilon[\mathbf{z}_0^{(N)}]$ (respectively $\varepsilon[\mathbf{z}_0^{(\infty)}]$). Actually, it will be necessary to take derivatives not just restricted to the subset $\widetilde{P}_1$ of $\widetilde{M}_1$. For this purpose, we note that by a simple scaling of our wave gravity Vlasov equations (57), (58), (62), we can first generalize the initial data $\in \widetilde{P}_1(\mathbb{R}^6)$ to those $\in \widetilde{M}_{1,+}(\mathbb{R}^6)$, and by the linearity of (62) in f given ψ together with the linearity of the map $f \mapsto \rho$, we can even allow the initial data for (62) to be a signed measure $\sigma_0 \in \widetilde{M}_1(\mathbb{R}^6)$; however, we always demand that $\operatorname{supp}(\sigma_0(\mathrm{d}x \times \mathbb{R}^3)) \subset B_R$. As to the initial data for (58), (62), we suitably extend the unique map $\mathbf{z}_0^{(N)} \mapsto \zeta[\mathbf{z}_0^{(N)}]$ to $\sigma_0 \mapsto \zeta[\sigma_0] \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ (slightly abusing notation), obeying (59), (60), and we write $\mathfrak{z}_t(\mathfrak{z}_0, \mathbf{z}_0^{(N)})$ by $\mathfrak{z}_t(\mathfrak{z}_0, \sigma_0)$. Note that $\mathfrak{z}_0$ itself does *not* depend on σ_0.

The (Gateaux) derivative $D^1 g(\sigma, .) : \mathbb{R}^6 \to \mathfrak{B}$ with respect to a finite measure $\sigma \in \widetilde{M}_1(\mathbb{R}^6)$ of a function $g(\sigma) \in \mathfrak{B}$ (any Banach space) is defined by the identity

$$\int D^1 g(\sigma, z)\nu(\mathrm{d}z) = \lim_{s \to 0} \frac{g(\sigma + s\nu) - g(\sigma)}{s}, \tag{170}$$

valid for all $\nu \in \widetilde{M}_1(\mathbb{R}^6)$; we here will restrict ν to satisfy $\operatorname{supp}(\nu_0(\mathrm{d}x \times \mathbb{R}^3)) \subset B_R$. Analogously one defines the higher derivatives $D^j g(\sigma, .)$ on $\mathbb{R}^{6j}$. To have a shorthand we write $\mathbf{z}^{(j)}$ for generic points in $\mathbb{R}^{6j}$; to achieve a more uniform notation in the presentation we will also write $\mathbf{z}^{(1)}$ for z in (170) and other first derivatives.

Next we define several auxiliary norms. Below, whenever we take the sup over $z_0, \zeta_0, \sigma_0, \mathbf{z}^{(j)}$, it is understood that $z_0 \in B_R \times \mathbb{R}^3$, that $\zeta_0 \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ obeying (59), (60), that $\sigma_0 \in \widetilde{M}_1$ has $\operatorname{supp}(\sigma_0(\mathrm{d}x \times \mathbb{R}^3)) \subset B_R$ and $|\sigma_0| \leq 1$, and that $\mathbf{z}^{(j)} \in (B_R \times \mathbb{R}^3)^j$. Thus, we define (noticing that $D^1 \nabla \psi_. = \nabla D^1 \psi_.$)

$$\|D^j z_t\|_{(\mathrm{u})} := \sup_{z_0, \sigma_0, \mathbf{z}^{(j)}} |D^j z_t(z_0, \sigma_0, \mathbf{z}^{(j)})| \tag{171}$$

$$\|D^j \psi_t\|_{(L)} := \sup_{\zeta_0, \sigma_0, \mathbf{z}^{(j)}} \|D^j \psi_t(\zeta_0, \sigma_0, \mathbf{z}^{(j)})\|_{L^2} \tag{172}$$

$$\|D^j \varpi_t\|_{(L)} := \sup_{\zeta_0, \sigma_0, \mathbf{z}^{(j)}} \|D^j \varpi_t(\zeta_0, \sigma_0, \mathbf{z}^{(j)})\|_{L^2} \tag{173}$$

$$\|D^j \psi_t\|_{(H)} := \sup_{\zeta_0, \sigma_0, \mathbf{z}^{(j)}} \|D^j \psi_t(\zeta_0, \sigma_0, \mathbf{z}^{(j)})\|_{\dot{H}^1}, \tag{174}$$

and we define $\|D^j \zeta_t\|_{(HL)}$ by setting

$$\|D^j \zeta_t\|_{(HL)}^2 := \|D^j \psi_t\|_{(H)}^2 + \|D^j \varpi_t\|_{(L)}^2 \tag{175}$$

PROPOSITION 4.10 *Given Cauchy data $\sigma_0 \in \widetilde{M}_1(\mathbb{R}^6)$ with $\mathrm{supp}\,(\sigma_0(\mathrm{d}x \times \mathbb{R}^3)) \subset B_R$ for (62), and associated with it Cauchy data $\zeta[\sigma_0] \in ((\dot{H}^1 \cap \dot{H}^2) \oplus L^2)(\mathbb{R}^3)$ for (57), (58), satisfying (59), (60), this Vlasovian Cauchy problem has a unique strong solution $(\sigma_\cdot, \zeta_\cdot^{(\sigma)}) \in (C_w^0 \cap C^1)(\mathbb{R}, \widetilde{M}_1 \oplus ((\dot{H}^1 \cap \dot{H}^2) \oplus L^2)(\mathbb{R}^3))$. This solution generates evolutions $(z_\cdot, \zeta_\cdot) = \mathfrak{z}_\cdot$ defined by (166)-(167) which $\forall k \in \mathbb{N}$ are k times continuously differentiable with respect to σ_0. Moreover, for all $\sigma_0 \mapsto \zeta[\sigma_0]$ such that there exist functions $B^j(.) \in C^0(\mathbb{R}^+)$, $j = 1, ..., k$, depending on σ_0 only through $\mathcal{E}(\sigma_0, \zeta[\sigma_0])$ and $|\sigma_0|$ such that $\|D^j\psi^{(\sigma)}(.,t',.)\|_{(H)} \leq B^j(t)$, there exist functions $B_\ell^j(.) \in C^0(\mathbb{R}^+)$, $j = 1, \ldots, k$, and $\ell = 1, 2, 3$, depending on σ_0 only through $\mathcal{E}(\sigma_0, \zeta[\sigma_0])$ and $|\sigma_0|$, such that for all $t \in \mathbb{R}$,*

$$\|D^j z_t\|_{(u)} \leq B_1^j(t)\,, \tag{176}$$

$$\|D^j \zeta_t\|_{(HL)} \leq B_2^j(t)\,, \tag{177}$$

$$\|D^j \psi_t\|_{(L)} \leq B_3^j(t)\,. \tag{178}$$

Proof. By inspection one verifies that our existence and uniqueness proof for the Vlasov model carries over to these more general data with only miniscule changes.

As to the existence and uniqueness of $D^j \mathfrak{z}_\cdot$, this follows in the standard way [Die60]. The first variation of the evolution equations yields $D^1 \mathfrak{z}_\cdot$ thus

$$D^1 q_t(z_0, \mathbf{z}^{(1)}) = \int_0^t \left[\frac{D^1 p_{t'}(z_0, \mathbf{z}^{(1)})}{(1 + |p_{t'}(z_0)|^2)^{1/2}} - \frac{p_{t'}(z_0) \otimes p_{t'}(z_0) \cdot D^1 p_{t'}(z_0, \mathbf{z}^{(1)})}{(1 + |p_{t'}(z_0)|^2)^{3/2}} \right] \mathrm{d}t' \tag{179}$$

$$D^1 p_t(z_0, \mathbf{z}^{(1)}) = \int_0^t \int \Big([\nabla \varrho(y - q_{t'}(z_0)) \cdot D^1 q_{t'}(z_0, \mathbf{z}^{(1)})] \nabla \psi^{(\sigma)}(y, t')$$
$$- \varrho(y - q_{t'}(z_0)) D^1 \nabla \psi^{(\sigma)}(y, t', \mathbf{z}^{(1)}) \Big) \mathrm{d}y \mathrm{d}t' \tag{180}$$

$$D^1 \psi_t(\zeta_0, \mathbf{z}^{(1)})(x) = \int_0^t (t - t') \fint_{\mathbb{S}^2} \left[\int [\nabla \varrho(x' - q_{t'}(z_0)) \cdot D^1 q_{t'}(z_0, \mathbf{z}^{(1)})] \sigma_0(\mathrm{d}z_0) \right.$$
$$\left. - \varrho\left(x' - q_{t'}(\mathbf{z}^{(1)})\right) \right] \mathrm{d}\Omega \mathrm{d}t' \tag{181}$$

$$D^1 \varpi_t(\zeta_0, \mathbf{z}^{(1)})(x) = \int_0^t \fint_{\mathbb{S}^2} \left[\int (D^1 q_{t'}(z_0, \mathbf{z}^{(1)}) \cdot \nabla)(1 + (t - t')\Omega \cdot \nabla) \varrho(x' - q_{t'}(z_0)) \sigma_0(\mathrm{d}z_0) \right.$$
$$\left. - [1 + (t - t')\Omega \cdot \nabla] \varrho(x' - q_{t'}(\mathbf{z}^{(1)})) \right] \mathrm{d}\Omega \mathrm{d}t' \tag{182}$$

$$D^1 \nabla \psi_t(\zeta_0, \mathbf{z}^{(1)})(x) = \int_0^t (t - t') \fint_{\mathbb{S}^2} \left[\int (D^1 q_{t'}(z_0, \mathbf{z}^{(1)}) \cdot \nabla) \nabla \varrho(x' - q_{t'}(z_0)) \sigma_0(\mathrm{d}z_0) \right.$$
$$\left. - \nabla \varrho(x' - q_{t'}(\mathbf{z}^{(1)})) \right] \mathrm{d}\Omega \mathrm{d}t'\,, \tag{183}$$

where once again we used the notation $x' = x + (t-t')\Omega$, with $\Omega \in \mathbb{S}^2$, and $\fint_{\mathbb{S}^2}$ to denote $\frac{1}{4\pi}\int_{\mathbb{S}^2}$; moreover, to shorten the presentation, we have not displayed the dependence on σ_0 of (z_t, ζ_t). Note also that in (181), (182), (183) we have transferred the time-dependence from $\sigma_{t'}$ to the adjoint time-dependence of $q_{t'}$ by change of variables.

Next, as to (181), one easily obtains the bound

$$\left\| \int_0^t (t-t') \fint_{\mathbb{S}^2} \varrho(.' - q_{t'}(\mathbf{z}^{(1)})) \mathrm{d}\Omega \mathrm{d}t' \right\|_{(L)} \leq \int_0^t (t-t') \sup_{\substack{|\sigma_0| \leq 1 \\ \mathbf{z}^{(1)} \in B_R \times \mathbb{R}^3}} \left\| \fint_{\mathbb{S}^2} \varrho(.' - q_{t'}(\mathbf{z}^{(1)})) \mathrm{d}\Omega \right\|_{L^2} \mathrm{d}t'$$

$$\leq \int_0^t (t-t') \|\varrho\|_{L^2} \, \mathrm{d}t' = t^2 \tfrac{1}{2} \|\varrho\|_{L^2}, \tag{184}$$

where $.' = . + (t-t')\Omega$, and similarly easily, using (171), one obtains

$$\left\| \int_0^t (t-t') \fint_{\mathbb{S}^2} \int \nabla\varrho(.' - q_{t'}(\mathbf{z}^{(1)})) \cdot D^1 q_{t'}(z_0, \mathbf{z}^{(1)}) \sigma_0(\mathrm{d}z_0) \mathrm{d}\Omega \mathrm{d}t' \right\|_{(L)}$$

$$\leq |\sigma_0| \int_0^t (t-t') \sup_{\substack{|\sigma_0| \leq 1 \\ \mathbf{z}^{(1)} \in B_R \times \mathbb{R}^3}} \left\| \fint_{\mathbb{S}^2} \int \nabla\varrho(.' - q_{t'}(\mathbf{z}^{(1)})) \cdot D^1 q_{t'}(z_0, \mathbf{z}^{(1)}) \sigma_0(\mathrm{d}z_0) \mathrm{d}\Omega \right\|_{L^2} \mathrm{d}t'$$

$$\leq |\sigma_0| \int_0^t (t-t') \|\nabla\varrho\|_{L^2} \|D^1 q_{t'}\|_{(\mathrm{u})} \mathrm{d}t'. \tag{185}$$

These bounds and similar ones for (182) and (183), and some obvious estimates for (179) and (180), yield

$$\|D^1 q_t\|_{(\mathrm{u})} \leq \int_0^t \|D^1 p_{t'}\|_{(\mathrm{u})} \mathrm{d}t' \tag{186}$$

$$\|D^1 p_t\|_{(\mathrm{u})} \leq \int_0^t \Big[\|\nabla\varrho\|_{L^2} \|\psi^{(\sigma)}(\,.\,, t')\|_{\dot{H}^1} \|D^1 q_{t'}\|_{(\mathrm{u})}$$

$$+ \|\varrho\|_{L^2} \|D^1 \psi^{(\sigma)}(., t', .)\|_{(H)} \Big] \mathrm{d}t' \tag{187}$$

$$\|D^1 \psi_t\|_{(H)} \leq t^2 \tfrac{1}{2} \|\nabla\varrho\|_{L^2} + \|\nabla^{\otimes 2}\varrho\|_{L^2} |\sigma_0| \int_0^t (t-t') \|D^1 q_{t'}\|_{(\mathrm{u})} \mathrm{d}t' \tag{188}$$

$$\|D^1 \varpi_t\|_{(L)} \leq t \|\varrho\|_{L^2} + t^2 \tfrac{1}{2} \|\nabla\varrho\|_{L^2}$$

$$+ |\sigma_0| \int_0^t \Big[\|\nabla\varrho\|_{L^2} + (t-t') \|\nabla^{\otimes 2}\varrho\|_{L^2} \Big] \|D^1 q_{t'}\|_{(\mathrm{u})} \mathrm{d}t' \tag{189}$$

$$\|D^1 \psi_t\|_{(L)} \leq t^2 \tfrac{1}{2} \|\varrho\|_{L^2} + \|\nabla\varrho\|_{L^2} |\sigma_0| \int_0^t (t-t') \|D^1 q_{t'}\|_{(\mathrm{u})} \mathrm{d}t', \tag{190}$$

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where $\|\nabla^{\otimes 2}\varrho\|_{L^2}^2 = \int |\nabla^{\otimes 2}\varrho|^2(x)\mathrm{d}x$, with $|M|$ the familiar Euclidean norm of a real symmetric 3×3 matrix as an element of $\mathbb{R}^9$.

We now recall that $\|\psi^{(\sigma)}(\,.\,,t')\|_{\dot{H}^1}$ is bounded uniformly for $t \in \mathbb{R}$ by a constant depending on σ_0 only through $\mathcal{E}(\sigma_0, \zeta[\sigma_0])$; cf. Remark 3.10.

Next, as a special case, assume first that $\zeta[\sigma_0] = \zeta_0$ independent of σ_0. In this case $\psi^{(\sigma)}(.,t) = \psi_t$, and the bounds (176), (177) for $j = 1$ follow from (179)–(182) by variants of the Gronwall lemma. The bound (178) for $j = 1$ then follows immediately. The bounds (176), (177) for general values of j follow by applying D^{j-1} to (179), (180), (181), (182), (183). In particular, the bound $\|D^j\psi^{(\sigma)}(.,t',.)\|_{(H)} \leq B^j(t)$, $j = 1, ..., k$, holds, with $B^j(.) \in C^0(\mathbb{R}^+)$ depending on σ_0 only through $\mathcal{E}(\sigma_0, \zeta[\sigma_0])$ and $|\sigma_0|$, for $\psi^{(\sigma)}(.,t) = \psi_t$ in this case.

Finally, the bounds clearly generalize to field data $\sigma_0 \mapsto \zeta[\sigma_0]$ for which there exist functions $B^j(.) \in C^0(\mathbb{R}^+)$, $j = 1, ..., k$, depending on σ_0 only through $\mathcal{E}(\sigma_0, \zeta[\sigma_0])$ and $|\sigma_0|$ such that $\|D^j\psi^{(\sigma)}(.,t',.)\|_{(H)} \leq B^j(t)$. That this hypothesis is legitimate we just showed, for its supposition is valid in particular when $\zeta[\sigma_0] = \zeta_0$.   Q.E.D.

REMARK 4.11 *Unfortunately we do not yet know how general the class of initial data $\sigma_0 \mapsto \zeta[\sigma_0]$ is which validate our hypothesis that $\|D^j\psi^{(\sigma)}(.,t',.)\|_{(H)} \leq B^j(t)$, $j = 1, ..., k$, with $B^j(.) \in C^0(\mathbb{R}^+)$ depending on σ_0 only through $\mathcal{E}(\sigma_0, \zeta[\sigma_0])$ and $|\sigma_0|$.*

Because of Remark 4.11, in the following we will restrict the initial conditions for the fields to the special case $\zeta[\mathbf{z}_0^{(N)}] = \zeta_0$ independent of $\mathbf{z}_0^{(N)}$; the case $N = \infty$ is included. In this vein, we may now analyze the limit $N \to \infty$ of the finite-N fluctuations

$$\Delta_z(t, z_0; \mathbf{z}_0^{(N)}, \mu_0) := \sqrt{N}[z_t(z_0; \mathbf{z}_0^{(N)}) - z_t(z_0, \mu_0)], \tag{191}$$

$$\Delta_\zeta(t, \zeta_0; \mathbf{z}_0^{(N)}, \mu_0) := \sqrt{N}[\zeta_t(\zeta_0; \mathbf{z}_0^{(N)}) - \zeta_t(\zeta_0, \mu_0)]. \tag{192}$$

We write $\Delta_\mathfrak{z} = (\Delta_z, \Delta_\zeta)$. Recall that by Theorem 4.2, if $\varepsilon[\mathbf{z}_0^{(N)}] \rightsquigarrow \mu_0$ as $N \to \infty$ and $\zeta[\mathbf{z}_0^{(N)}] = \zeta_0^{(\infty)} \equiv \zeta_0$ for all N, then

$$\varepsilon[\mathbf{z}_t^{(N)}] \quad \rightsquigarrow \quad \mu_t \tag{193}$$

$$\psi_t(\zeta_0; \mathbf{z}_0^{(N)}) \quad \xrightarrow{\dot{H}^1} \quad \psi_t(\zeta_0; \mu_0) \tag{194}$$

$$\varpi_t(\zeta_0; \mathbf{z}_0^{(N)}) \quad \xrightarrow{L^2} \quad \varpi_t(\zeta_0; \mu_0) \tag{195}$$

and, due to Corollary 3.7,

$$|z_t(z_0, \mathbf{z}_0^{(N)}) - z_t(z_0, \mu_0)| \longrightarrow 0. \tag{196}$$

We now are ready to state the central limit theorem.

THEOREM 4.12 *For $\mu_0 \in P_1(\mathbb{R}^6)$, define the sequence of particle product measures $\mu_0^{\times N}(\mathrm{d}\mathbf{z}_0^{(N)}) = \prod_{i=1}^{N} \mu_0(\mathrm{d}z_i^{(N)}(0))$, and consider field initial data $\zeta_0 \in (\dot{H}^1 \cap \dot{H}^2)(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. For any $K \geq 1$ and $1 \leq k \leq K$, let $Z_k \in \Gamma^{(1)}$, $z_k \in \mathbb{R}^6$, $t_k \in \mathbb{R}$ and define $\vec{Z} = (Z_1, \ldots, Z_K) \in (\Gamma^{(1)})^K$, $\vec{z} = (z_1, \ldots, z_k) \in \mathbb{R}^{6K}$, and $\vec{t} = (t_1, \ldots, t_K) \in \mathbb{R}^K$. Moreover, let $\vec{\Delta}_{\mathfrak{z}}(\vec{t}, \vec{z}, \zeta_0; \mathbf{z}_0^{(N)}) \in (\Gamma^{(1)})^K$ have k-th component $\Delta_{\mathfrak{z}}(t_k, z_k, \zeta_0; \mathbf{z}_0^{(N)})$. Then*

$$\lim_{N \to \infty} \int e^{\mathrm{i}\langle\langle \vec{Z} | \vec{\Delta}_{\mathfrak{z}}(\vec{t}, \vec{z}, \zeta_0; \mathbf{z}_0^{(N)})\rangle\rangle} \mu_0^{\times N}(\mathrm{d}\mathbf{z}_0^{(N)}) = e^{-\frac{1}{2}\langle \vec{Z} | \overleftrightarrow{Q} | \vec{Z}\rangle} \tag{197}$$

where $\langle\,.\,|\,.\,\rangle$ is the scalar product in the Hilbert space $(\Gamma^{(1)})^K$ (i.e. the sum over K scalar products $\langle\,.\,,\,.\,\rangle$ in $\Gamma^{(1)}$ indexed by $_k$), and the operator $\overleftrightarrow{Q}$ has k, k' component

$$Q_{k,k'} = \int \bigotimes_{\kappa \in \{k,k'\}} D^1 \mathfrak{z}_{t_\kappa}(z_\kappa, \zeta_0, \mu_0, \mathbf{z}^{(1)}) \mu_0(\mathrm{d}\mathbf{z}^{(1)})$$

$$- \bigotimes_{\kappa \in \{k,k'\}} \int D^1 \mathfrak{z}_{t_\kappa}(z_\kappa, \zeta_0, \mu_0, \mathbf{z}_\kappa^{(1)}) \mu_0(\mathrm{d}\mathbf{z}_\kappa^{(1)}) . \tag{198}$$

For $\vec{t}$ and $\vec{Z}$ in bounded sets, the convergence is uniform in $\vec{z}$.
The stochastic process $\eta = \lim_{N \to \infty} \Delta_{\mathfrak{z}}$ on $\Gamma^{(1)}$, with vanishing expectation and covariance (198), can be represented as

$$\eta_t(z_0, \mu_0) = \int D^1 \mathfrak{z}_t(z_0, \mu_0, \mathbf{z}^{(1)}) \varphi(\mu_0, \mathrm{d}\mathbf{z}^{(1)}) \tag{199}$$

and satisfies the equations obtained integrating (179), (180), (181), (182) with respect to $\varphi(\mu_0, \mathrm{d}\mathbf{z}^{(1)})$ with initial conditions $\eta_0(z_0, \mu_0) = 0$. Here, the random measure $\varphi(\mu, \mathrm{d}\mathbf{z}^{(1)}) \in M(\mathbb{R}^6)$ with Gaussian law is defined by

$$\mathbb{E}_{\mu_0^{\times N}} \varphi(\mu_0, \triangle_1) = 0 , \tag{200}$$

$$\mathbb{E}_{\mu_0^{\times N}}[\varphi(\mu_0, \triangle_1)\varphi(\mu_0, \triangle_2)] = \mu_0(\triangle_1 \cap \triangle_2) - \mu_0(\triangle_1)\mu_0(\triangle_2) , \tag{201}$$

for measurable $\triangle_1, \triangle_2 \subset \mathbb{R}^6$.

Proof. Apart from the different kind of convergence for the fluctuations of the potential, the proof is analogous to the one in [BrHe77]. We carry out some calculations to clarify the procedure. Let $\nu[\mathbf{z}_0^{(N)}] := \varepsilon[\mathbf{z}_0^{(N)}] - \mu_0$. The following calculations are valid whenever $D^1 \mathfrak{z}$. exists.

Obviously,

$$\mathfrak{z}_t(z_0, \varepsilon_{\mathbf{z}_0^{(N)}}) - \mathfrak{z}_t(z_0, \mu_0) = \int_0^1 \int D^1 \mathfrak{z}_t(z_0, \zeta_0, \mu_0 + r\nu[\mathbf{z}_0^{(N)}], \mathbf{z}^{(1)})\nu[\mathbf{z}_0^{(N)}](\mathrm{d}\mathbf{z}^{(1)})\mathrm{d}r. \quad (202)$$

We now define

$$\Xi_k^{(1)}(\sigma, \mathbf{z}^{(1)}) = \sqrt{N} D^1 \mathfrak{z}_{t_k}(z_k, \zeta_0, \sigma, \mathbf{z}^{(1)}), \quad (203)$$

$$\Xi_k^{(2)}(\sigma, \mathbf{z}^{(2)}) = \sqrt{N} D^2 \mathfrak{z}_{t_k}(z_k, \zeta_0, \sigma, \mathbf{z}^{(2)}), \quad (204)$$

$$\xi_k^{(1)}(\sigma) = \int \Xi_k^{(1)}(\sigma, \mathbf{z}^{(1)})\nu[\mathbf{z}_0^{(N)}](\mathrm{d}\mathbf{z}^{(1)}), \quad (205)$$

$$\xi_k^{(2)}(\sigma) = \iint \Xi_k^{(2)}(\sigma, \mathbf{z}^{(2)})\nu[\mathbf{z}_0^{(N)}]^{\times 2}(\mathrm{d}\mathbf{z}^{(2)}), \quad (206)$$

and group the corresponding K components into the vectors $\vec{\Xi}^{(1)}$, $\vec{\Xi}^{(2)}$, $\vec{\xi}^{(1)}$ and $\vec{\xi}^{(2)}$. We also define

$$\tilde{\xi}_{kj}(\sigma) = \frac{1}{\sqrt{N}}\left(\Xi_k^{(1)}(\sigma, z_j^{(N)}(0)) - \int \Xi_k^{(1)}(\sigma, \mathbf{z}^{(1)})\sigma(\mathrm{d}\mathbf{z}^{(1)})\right) \quad (207)$$

where $z_j^{(N)}(0)$ is the j-th component of $\mathbf{z}_0^{(N)}$, and note that

$$\xi_k^{(1)} = \tfrac{1}{\sqrt{N}}\sum_{j=1}^N \tilde{\xi}_{kj} \quad (208)$$

and that

$$\mathbb{E}_{\mu_0^{\times N}}(\tilde{\xi}_{kj}) = 0, \quad (209)$$

$$\mathbb{E}_{\mu_0^{\times N}}(\tilde{\xi}_{kj}\tilde{\xi}_{k'j'}) = Q_{k,k'}\delta_{j,j'}, \quad (210)$$

where $\delta_{j,j'}$ is the Kronecker symbol. Hence the central limit theorem applies to $\xi_k^{(1)}$.

We may now write

$$e^{i\langle \vec{Z}|\vec{\Delta}_{\mathfrak{z}}\rangle} = e^{i\langle \vec{Z}|\vec{\xi}^{(1)}\rangle} + \int_0^1 \tfrac{\mathrm{d}}{\mathrm{d}s}\exp\left(i\int_0^1 \langle \vec{Z}|\vec{\xi}^{(1)}(\mu_0 + rs\nu[\mathbf{z}_0^{(N)}])\rangle \mathrm{d}r\right)\mathrm{d}s. \quad (211)$$

When $N \to \infty$, the expectation of the first term in the right-hand side converges to the right-hand side of (197).

The integrand in the second term becomes, after differentiation w.r.t. s,

$$\mathrm{i}\int_0^1 \langle \vec{Z}|\vec{\xi}^{(2)}(\mu_0 + rs\nu[\mathbf{z}_0^{(N)}])\rangle r\mathrm{d}r \exp\left(\mathrm{i}\int_0^1 \langle \vec{Z}|\vec{\xi}^{(1)}(\mu_0 + r's\nu[\mathbf{z}_0^{(N)}])\rangle \mathrm{d}r'\right)$$

$$= \frac{\mathrm{i}}{N\sqrt{N}} \sum_{i,j=1}^N \Big[\mathcal{G}(z_i^{(N)}(0), z_j^{(N)}(0), \varepsilon[\mathbf{z}_0^{(N)}]) - \int \mathcal{G}(z_i^{(N)}(0), \mathbf{z}^{(1)}, \varepsilon[\mathbf{z}_0^{(N)}])\mu_0(\mathrm{d}\mathbf{z}^{(1)})$$

$$- \int \mathcal{G}(\mathbf{z}^{(1)}, z_j^{(N)}(0), \varepsilon[\mathbf{z}_0^{(N)}])\mu_0(\mathrm{d}\mathbf{z}^{(1)}) + \iint \mathcal{G}(\mathbf{z}^{(2)}, \varepsilon[\mathbf{z}_0^{(N)}])\mu_0^{\times 2}(\mathrm{d}\mathbf{z}^{(2)})\Big]$$

$$\exp\left(\frac{\mathrm{i}}{\sqrt{N}}\sum_{i'=1}^N \Big[\mathcal{F}(z_{i'}^{(N)}(0), \varepsilon[\mathbf{z}_0^{(N)}]) - \int \mathcal{F}(\tilde{\mathbf{z}}^{(1)}, \varepsilon[\mathbf{z}_0^{(N)}])\mu_0(\mathrm{d}\tilde{\mathbf{z}}^{(1)})\Big]\right) \qquad (212)$$

where (for given $\vec{Z}$, $\vec{z}$, $\vec{t}$, ζ_0, μ_0, s)

$$\sqrt{N}\mathcal{F}(\mathbf{z}^{(1)}, \sigma_0) = \int_0^1 \langle \vec{Z}|\vec{\Xi}^{(1)}(\mu_0 + rs(\sigma_0 - \mu_0), \mathbf{z}^{(1)})\rangle \mathrm{d}r\,, \qquad (213)$$

$$\sqrt{N}\mathcal{G}(\mathbf{z}^{(2)}, \sigma_0) = \int_0^1 \langle \vec{Z}\vec{\Xi}^{(2)}(\mu_0 + rs(\sigma_0 - \mu_0), \mathbf{z}^{(2)})\rangle r\mathrm{d}r\,. \qquad (214)$$

The regularization ensures that $\mathcal{G}$ and $\mathcal{F}$ are differentiable to any order with respect to σ_0, with bounded derivatives in the sense of Proposition 4.10. We may then evaluate the size of the expectation of (212) in the following way.

Expression (212) can be split in two components : the "diagonal" part, which is obtained from the terms in the sum such that $i = j$, gives with trivial estimates a contribution of order $N^{-1/2}$ to the expectation of (212) ; an estimate of the size of the "non-diagonal" component needs some more manipulation.

Consider the measure (positive with mass $1 - 2/N$)

$$\mu^{ij}[\mathbf{z}_0^{(N)}] = \frac{1}{N}\sum_{\substack{k=1 \\ k\neq i,j}}^N \delta_{z_k^{(N)}(0)} = \varepsilon[\mathbf{z}_0^{(N)}] - \frac{1}{N}(\delta_{z_i^{(N)}(0)} + \delta_{z_j^{(N)}(0)})\,. \qquad (215)$$

Given the result in Proposition 4.10, we may write

$$\mathcal{F}(\mathbf{z}^{(1)}, \varepsilon[\mathbf{z}_0^{(N)}]) = \mathcal{F}(\mathbf{z}^{(1)}, \mu^{ij}[\mathbf{z}_0^{(N)}]) + \frac{1}{N}\sum_{k\in\{i,j\}} D^1\mathcal{F}(\mathbf{z}^{(1)}, \mu^{ij}[\mathbf{z}_0^{(N)}], z_k^{(N)}(0)) + \mathrm{O}(\tfrac{1}{N^2})\,,$$

$$\mathcal{G}(\mathbf{z}^{(2)}, \varepsilon[\mathbf{z}_0^{(N)}]) = \mathcal{G}(\mathbf{z}^{(2)}, \mu^{ij}[\mathbf{z}_0^{(N)}]) + \frac{1}{N}\sum_{k\in\{i,j\}} D^1\mathcal{G}(\tilde{\mathbf{z}}^{(1)}, \mu^{ij}[\mathbf{z}_0^{(N)}], z_k^{(N)}(0)) + \mathrm{O}(\tfrac{1}{N^2})\,.$$

For a given pair $\{i, j\}$, we then obtain

$$\exp\left(\frac{\mathrm{i}}{\sqrt{N}}\sum_{n=1}^{N}\left[\mathcal{F}(z_n^{(N)}(0), \varepsilon[\mathbf{z}_0^{(N)}]) - \int \mathcal{F}(\mathbf{z}^{(1)}, \varepsilon[\mathbf{z}_0^{(N)}])\mu_0(\mathrm{d}\mathbf{z}^{(1)})\right]\right)$$
$$= \left[1 + \frac{\mathrm{i}}{\sqrt{N}}\left[\mathcal{F}_1(\hat{z}_i^{(N)}(0)) + \mathcal{F}_1(\hat{z}_j^{(N)}(0))\right] + \mathrm{O}\left(\frac{1}{N}\right)\right]\mathcal{F}_2(\hat{z}_i^{(N)}(0), \hat{z}_j^{(N)}(0)) \quad (216)$$

where we denote by $g(\hat{x})$ a function such that $\partial_x g = 0$, which is bounded according to Proposition 4.10 and need not be further specified. Moreover,

$$\mathcal{G}(z_i^{(N)}(0), z_j^{(N)}(0), \varepsilon[\mathbf{z}_0^{(N)}]) - \int \mathcal{G}(z_i^{(N)}(0), \tilde{\mathbf{z}}^{(1)}, \varepsilon[\mathbf{z}_0^{(N)}])\mu_0(\mathrm{d}\tilde{\mathbf{z}}^{(1)})$$
$$- \int \mathcal{G}(\mathbf{z}^{(1)}, z_j^{(N)}(0), \varepsilon[\mathbf{z}_0^{(N)}])\mu_0(\mathrm{d}\mathbf{z}^{(1)})$$
$$+ \iint \mathcal{G}(\mathbf{z}^{(2)}, \varepsilon[\mathbf{z}_0^{(N)}])\mu_0^{\times 2}(\mathrm{d}\mathbf{z}^{(2)}) = \mathcal{D}_{ij} + \mathrm{O}\left(\frac{1}{N}\right) \quad (217)$$

where

$$\mathcal{D}_{ij} = \mathcal{G}(z_i^{(N)}(0), z_j^{(N)}(0), \mu^{ij}[\mathbf{z}_0^{(N)}]) - \int \mathcal{G}(z_i^{(N)}(0), \tilde{\mathbf{z}}^{(1)}, \mu^{ij}[\mathbf{z}_0^{(N)}])\mu_0(\mathrm{d}\tilde{\mathbf{z}}^{(1)})$$
$$- \int \mathcal{G}(\mathbf{z}^{(1)}, z_j^{(N)}(0), \mu^{ij}[\mathbf{z}_0^{(N)}])\mu_0(\mathrm{d}\mathbf{z}^{(1)})$$
$$+ \iint \mathcal{G}(\mathbf{z}^{(2)}, \mu^{ij}[\mathbf{z}_0^{(N)}])]\mu_0^{\times 2}(\mathrm{d}\mathbf{z}^{(2)})$$

so that

$$\mathbb{E}_{\mu_0^{\times N}}[\mathcal{D}_{ij}] = 0, \quad (218)$$
$$\mathbb{E}_{\mu_0^{\times N}}[\mathcal{D}_{ij}g(\hat{z}_k^{(N)}(0))] = 0, \quad (219)$$

for any bounded function $g(\hat{z}_k^{(N)}(0))$ with $k \in \{i, j\}$. The expectation of the non-diagonal component of (212) is then given by

$$\frac{\mathrm{i}}{N\sqrt{N}}\sum_{\substack{i,j=1 \\ i \neq j}}^{N}\int\left[\mathcal{D}_{ij} + \mathrm{O}\left(\frac{1}{N}\right)\right]\left[1 + \frac{\mathrm{i}}{\sqrt{N}}\left(\mathcal{F}_1(\hat{z}_i^{(N)}(0)) + \mathcal{F}_1(\hat{z}_j^{(N)}(0))\right) + \mathrm{O}\left(\frac{1}{N}\right)\right] \times$$

$$\times \mathcal{F}_2\left(\hat{z}_i^{(N)}(0), \hat{z}_j^{(N)}(0)\right)\mu_0^{\times N}(\mathrm{d}\mathbf{z}_0^{(N)}) = \frac{N(N-1)}{N\sqrt{N}}\mathrm{O}\left(\frac{1}{N}\right) \quad (220)$$

so that the second term in (211) is $\mathrm{O}(N^{-1/2})$.

The identification of the limit stochastic process is obtained from (179)-(180)-(181)-(182). Q.E.D.

Appendix

A.1 Nested modes of convergence of probability measures

A certain frustration about the absence of an authoritative survey of the relationships of various important notions of convergence that are used in the probability literature has already been expressed [GiSu02], where that gap has been filled to some extent. Unfortunately, [GiSu02] does not cover all our needs. Furthermore, when addressing a mixed readership of mathematical physicists, analysts and probabilists, the frustration can get compounded by the various 'competing' terminologies and notations that are in use in these areas of activity. In view of this, it seems advisable to be more explicit about how the notions of convergence that we use. The following general notions hold (and are formulated) for any dimension $d \geq 1$.

We recall that, if $\{\mu_n\}_{n \in \mathbb{N}}$ is a sequence of Borel probability measures on $\mathbb{R}^d$ and $\mu \in P(\mathbb{R}^d)$, too, and if $\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$ for every bounded continuous function $f \in C_b^0(\mathbb{R}^d)$, then one says that μ_n converges to μ *in law*,[12] written $\mu_n \overset{\mathcal{L}}{\longrightarrow} \mu$; see p. 292 of [Dud02]. Clearly, since $C_0^0(\mathbb{R}^d) \subset C_b^0(\mathbb{R}^d)$, convergence in law $\mu_n \overset{\mathcal{L}}{\longrightarrow} \mu$ implies vague convergence $\mu_n \rightharpoonup \mu$. Moreover, convergence in law $\mu_n \overset{\mathcal{L}}{\longrightarrow} \mu$ is equivalent to *convergence in probability* of the underlying family of random variables having laws μ_n to a random variable with law μ, a notion we need for our law of large numbers.

Convergence in law can be metrized as follows. Let $C_b^{0,\alpha}(\mathbb{R}^d)$ denote the subset of the bounded continuous functions on $\mathbb{R}^d$ which are also Hölder continuous with exponent $\alpha \in (0,1]$. Now $C_b^{0,\alpha}(\mathbb{R}^d)$ is not a closed subspace of $C_b^0(\mathbb{R}^d)$ w.r.t. $\| \cdot \|_{\mathrm{u}}$, but

$$\|g\|_{\mathrm{u},\alpha} \equiv \max \left\{ \|g\|_{\mathrm{u}} , \mathrm{H\ddot{o}l}_\alpha(g) \right\} , \tag{221}$$

where

$$\mathrm{H\ddot{o}l}_\alpha(g) \equiv \sup_{\xi \neq \xi' \in \mathbb{R}^d} \frac{|g(\xi) - g(\xi')|}{|\xi - \xi'|^\alpha} \tag{222}$$

is the α-Hölder seminorm of g, turns $C_b^{0,\alpha}(\mathbb{R}^d)$ into a (non-separable) Banach space. The positive cone in $C_b^{0,\alpha}(\mathbb{R}^d)$ is denoted by $C_{b,+}^{0,\alpha}(\mathbb{R}^d)$. If the suffix $_b$ is replaced by the suffix $_0$, we mean the corresponding subsets of these functions that vanish at infinity. In much of what follows, we will need $C_b^{0,1}(\mathbb{R}^d)$, the space of bounded Lipschitz functions on $\mathbb{R}^d$, and we write[13] $\mathrm{Lip}\,(g)$ for $\mathrm{H\ddot{o}l}_\alpha(g)$ when $\alpha = 1$.[14]

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[12]In the probability literature, convergence in law is usually called "weak convergence" of probability measures; however, this notion generally differs from the analysts' notion of weak convergence on M.

[13]Since $\mathrm{Lip}\,(\,.\,)$ is a seminorm, we prefer this notation over $\| \cdot \|_{\mathrm{L}}$, which is also in use in the literature.

[14]We recall that if $g \in C^1(\mathbb{R}^d)$, then $\mathrm{Lip}\,(g) = \sup_{x \in \mathbb{R}^d} |\nabla g(x)|$.

Now let $\mu_1 \in P(\mathbb{R}^d)$ and $\mu_2 \in P(\mathbb{R}^d)$ be two Borel probability measures on $\mathbb{R}^d$. We define the dual bounded-Lipschitz distance between μ_1 and μ_2 as[15]

$$\text{dist}_{\text{bL}^*}(\mu_1, \mu_2) := \sup_{g \in C_b^{0,1}(\mathbb{R}^d)} \left\{ \left| \int g \, \mathrm{d}(\mu_1 - \mu_2) \right| : \|g\|_{\text{u},1} \leq 1 \right\}. \tag{223}$$

Our dual bounded-Lipschitz distance, though not identical to, is equivalent to the Fortet–Mourier β-distance (p.395 of [Dud02]), which instead of $\|g\|_{\text{u},1} \leq 1$ works with the equivalent condition $\|g\|_{\text{u}} + \text{Lip}(g) \leq 1$. Therefore, by Proposition 11.3.2 of [Dud02], $\text{dist}_{\text{bL}^*}(\,,\,)$ is a metric on the convex set $P(\mathbb{R}^d)$, and by Corollary 11.5.5 of [Dud02], $P(\mathbb{R}^d)$ is complete for $\text{dist}_{\text{bL}^*}(\,,\,)$. Furthermore, by Theorem 11.3.3 of [Dud02], if $\{\mu_n\}_{n \in \mathbb{N}}$ is a sequence of Borel probability measures on $\mathbb{R}^d$, and $\mu \in P(\mathbb{R}^d)$, too, then $\text{dist}_{\text{bL}^*}(\mu_n, \mu) \to 0$ as $n \to \infty$ is equivalent to $\mu_n \overset{\mathcal{L}}{\longrightarrow} \mu$ as $n \to \infty$. Hence, $\text{dist}_{\text{bL}^*}(\,,\,)$ metrizes convergence in law of the Borel probability measure on $\mathbb{R}^d$.

Our dual bounded-Lipschitz distance $\text{dist}_{\text{bL}^*}(\,,\,)$ is equivalent, but not identical, to the distance obtained by restricting g to $C_{b,+}^{0,\alpha}(\mathbb{R}^d)$, here denoted $\text{d}_{\text{bL}^*}(\,,\,)$ (following [Spo91], Def. 2.2; actually, Spohn writes $\text{d}_{\text{bL}}(\,,\,)$, but we here better keep the *). Clearly, $\text{dist}_{\text{bL}^*}(\mu_n, \mu) \to 0$ implies $\text{d}_{\text{bL}^*}(\mu_n, \mu) \to 0$. The converse of this follows from three simple observations: first, the integral on the r.h.s. of (223) is invariant under $g \to g + \|g\|_{\text{u}}$, so that in our definition of $\text{dist}_{\text{bL}^*}(\,,\,)$ we can replace $C_b^{0,1}(\mathbb{R}^d)$ by $C_{b,+}^{0,1}(\mathbb{R}^d)$ and simultaneously replace the condition $\|g\|_{\text{u},1} \leq 1$ with the condition $\max\{\frac{1}{2}\|g\|_{\text{u}}, \text{Lip}(g)\} \leq 1$; second, $\{g \in C_{b,+}^{0,1}(\mathbb{R}^d) : \|g\|_{\text{u}} \leq 2, \text{Lip}(g) \leq 1\}$ is a strict subset of $\{g \in C_{b,+}^{0,1}(\mathbb{R}^d) : \|g\|_{\text{u}} \leq 2, \text{Lip}(g) \leq 2\}$; third, the simple scaling $g \to 2g$ reveals that the sup of $\left| \int g \, \mathrm{d}(\mu_1 - \mu_2) \right|$ over $\{g \in C_{b,+}^{0,1}(\mathbb{R}^d) : \|g\|_{\text{u}} \leq 2, \text{Lip}(g) \leq 2\}$ is twice the sup of $\left| \int g \, \mathrm{d}(\mu_1 - \mu_2) \right|$ over $\{g \in C_{b,+}^{0,1}(\mathbb{R}^d) : \|g\|_{\text{u}} \leq 1, \text{Lip}(g) \leq 1\}$. These three facts together imply that $\text{dist}_{\text{bL}^*}(\mu_1, \mu_2) \leq 2\text{d}_{\text{bL}^*}(\mu_1, \mu_2)$, and this means that $\text{dist}_{\text{bL}^*}(\mu_n, \mu) \to 0$ whenever $\text{d}_{\text{bL}^*}(\mu_n, \mu) \to 0$.

Recall that the general Kantorovich–Rubinstein distance[16] is defined as

$$\text{dist}_{\text{KRc}}(\mu_1, \mu_2) := \inf_{\mu \in P_c(\mathbb{R}^{2d}|\mu_1,\mu_2)} \left\{ \int \text{cost}(\xi_1, \xi_2) \mu(\mathrm{d}\xi_1 \mathrm{d}\xi_2) \right\}, \tag{224}$$

where $\text{cost}(\xi, \xi') = \text{dist}_{\text{KRc}}(\delta_\xi, \delta_{\xi'})$ for $\xi, \xi' \in \mathbb{R}^d$ is the "cost (per transport unit) function," and where $P_c(\mathbb{R}^{2d}|\mu_1,\mu_2)$ is the set of Borel probability measures μ on $\mathbb{R}^d \times \mathbb{R}^d$

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[15]The * at $\text{dist}_{\text{bL}^*}(\,,\,)$ refers to the Kantorovich–Rubinstein duality theorems; see below.

[16]Also associated with the names of Monge and Wasserstein.

satisfying $\mu(\mathrm{d}\xi_1 \times \mathbb{R}^d) = \mu_1(\mathrm{d}\xi_1)$ and $\mu(\mathbb{R}^d \times \mathrm{d}\xi_2) = \mu_2(\mathrm{d}\xi_2)$, with μ_1 and μ_2 satisfying $\int \mathrm{cost}(\xi_1, \xi)\mu_1(\mathrm{d}\xi_1) < \infty$ and $\int \mathrm{cost}(\xi, \xi_2)\mu_2(\mathrm{d}\xi_2) < \infty$ for some $\xi \in \mathbb{R}^d$.

By the Kantorovich–Rubinstein theorem ([Dud02], Theorem 11.8.2), $\mathrm{dist}_{\mathrm{bL}^*}(\mu_1, \mu_2)$ is identical to the Kantorovich–Rubinstein distance for $\mathrm{cost}(\xi_1, \xi_2) = \min\{2, |\xi_1 - \xi_2|\}$. Incidentally, $\mathrm{cost}(\xi_1, \xi_2) = \min\{1, |\xi_1 - \xi_2|\}$ is the cost function for the particular Kantorovich–Rubinstein distance identical to $\mathrm{d}_{\mathrm{bL}^*}(\ ,\ )$. The dual bounded-Lipschitz distance $(\mathrm{d}_{\mathrm{bL}^*})$ is used in [NeWi74, BrHe77, Neu85, Spo91] and [FiEl98].

However, if one is only interested, as we are, in the subset $P_1(\mathbb{R}^d) \subset P(\mathbb{R}^d)$, it is rather prudent to work with the dual Lipschitz distance in $P_1(\mathbb{R}^d)$, given by

$$\mathrm{dist}_{\mathrm{L}^*}(\mu_1, \mu_2) := \sup_{g \in C^{0,1}(\mathbb{R}^d)} \left\{ \left| \int g \, \mathrm{d}(\mu_1 - \mu_2) \right| : \mathrm{Lip}\,(g) \le 1 \right\}, \qquad (225)$$

which is identical with the standard[17] Kantorovich–Rubinstein distance, given by

$$\mathrm{dist}_{\mathrm{KR}}(\mu_1, \mu_2) := \inf_{\mu \in P_1(\mathbb{R}^{2d}|\mu_1, \mu_2)} \left\{ \int |\xi_1 - \xi_2| \mu(\mathrm{d}\xi_1 \mathrm{d}\xi_2) \right\}, \qquad (226)$$

where $P_1(\mathbb{R}^{2d}|\mu_1, \mu_2)$ is the set of Borel probability measures μ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $\mu(\mathrm{d}\xi_1 \times \mathbb{R}^d) = \mu_1(\mathrm{d}\xi_1) \in P_1(\mathbb{R}^d)$ and $\mu(\mathbb{R}^d \times \mathrm{d}\xi_2) = \mu_2(\mathrm{d}\xi_2) \in P_1(\mathbb{R}^d)$. We write $\mu_n \rightsquigarrow \mu$ if $\mathrm{dist}_{\mathrm{L}^*}(\mu_n, \mu) \to 0$. Clearly, $\mathrm{dist}_{\mathrm{L}^*}(\mu_n, \mu) \to 0$ implies[18] $\mathrm{dist}_{\mathrm{bL}^*}(\mu_n, \mu) \to 0$.

We note that the metric $\mathrm{dist}_{\mathrm{L}^*}(\,.\,,\,.\,)$ defines a norm $\|\,.\,\|_{\mathrm{L}^*}$ on $(P_1 - P_1) \subset M$ by[19] $\|\sigma\|_{\mathrm{L}^*} := \mathrm{dist}_{\mathrm{L}^*}(\sigma_+, \sigma_-)$. This definition extends identically to $\lambda(P_1 - P_1)$ for any $\lambda \in \mathbb{R}$. To extend $\|\,.\,\|_{\mathrm{L}^*}$ to the linear span of P_1 for $\sigma \in \mathrm{lsp}\, P_1$ we define

$$\|\sigma\|_{\widetilde{\mathrm{L}^*}} := \mathrm{dist}_{\mathrm{L}^*}\big((\sigma - \sigma(\mathbb{R}^d)\tilde{\mu})_+, (\sigma - \sigma(\mathbb{R}^d)\tilde{\mu})_-\big) + |\sigma(\mathbb{R}^d)| \qquad (227)$$

where $\tilde{\mu} \in P_1(\mathbb{R}^d)$ is arbitrary but fixed; e.g. $\tilde{\mu} = \delta_0$. Clearly, for $\sigma \in P_1 - P_1$, such that $\sigma(\mathbb{R}^d) = 0$, (227) reduces to $\|\sigma\|_{\widetilde{\mathrm{L}^*}} = \mathrm{dist}_{\mathrm{L}^*}(\sigma_+, \sigma_-)$, i.e. $\|\sigma\|_{\widetilde{\mathrm{L}^*}} = \|\sigma\|_{\mathrm{L}^*}$ whenever $\sigma(\mathbb{R}^d) = 0$. It is straightforward to verify that $\|\,.\,\|_{\widetilde{\mathrm{L}^*}}$ is a norm on $\mathrm{lsp}\, P_1$. The completion of the linear span of $P_1(\mathbb{R}^d)$ w.r.t. (227), denoted $\widetilde{M}_1(\mathbb{R}^d)$, is a Banach space with norm $\|\,.\,\|_{\widetilde{\mathrm{L}^*}}$ given in (227). We write $\widetilde{P}_1(\mathbb{R}^d)$ for $P_1(\mathbb{R}^d) \hookrightarrow \widetilde{M}_1(\mathbb{R}^d)$.

[17]The word "standard" refers to the custom in the probability community that, by default, the cost function is identified with the metric of the underlying complete metric space on which the Borel probability measures are defined; in standard Euclidean $\mathbb{R}^d$ this gives $\mathrm{cost}(\xi_1, \xi_2) = |\xi_1 - \xi_2|$.

[18]The converse is not true. In particular, Dudley gives the following counterexample for $d = 1$: $\mu_n = (1 - n^{-1})\delta_0 + n^{-1}\delta_n$ and $\mu = \delta_0$, for which $\mathrm{dist}_{\mathrm{L}^*}(\mu_n, \mu) = 1$ while $\mathrm{dist}_{\mathrm{bL}^*}(\mu_n, \mu) \le 2n^{-1} \downarrow 0$.

[19]In particular, if $\sigma = \mu_1 - \mu_2$ with $\mu_1, \mu_2 \in P_1$, then $\mathrm{dist}_{\mathrm{L}^*}(\sigma_+, \sigma_-) = \mathrm{dist}_{\mathrm{L}^*}(\mu_1, \mu_2)$; note, however, that generally $\mu_1 \ne (\mu_1 - \mu_2)_+$ and $\mu_2 \ne (\mu_1 - \mu_2)_-$.

A.2 The second order variant of the Gronwall lemma

The standard Gronwall lemma provides a simple upper bound on a function $t \mapsto u(t)$ satisfying the first order differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}u \leq f(t)u + g(t) \tag{228}$$

for all $t \in \mathbb{R}_+$, with $u(0) = u_0 > 0$, and with $f(t)$ and $g(t)$ given positive continuous functions; namely, with the help of an integrating factor one finds right away that u is bounded by

$$u(t) \leq u_0 \exp\left(\int_0^t f(\tau)\mathrm{d}\tau\right) + \int_0^t \exp\left(\int_\tau^t f(\tilde{\tau})\mathrm{d}\tilde{\tau}\right) g(\tau)\mathrm{d}\tau. \tag{229}$$

In particular, if $f(t) \equiv \gamma > 0$ is a constant, then

$$u(t) \leq u_0 \exp(\gamma t) + \int_0^t \exp[\gamma(t-\tau)]g(\tau)\mathrm{d}\tau. \tag{230}$$

However, (229) does not suit our purposes; instead, we need the following second order variant of (229):

LEMMA A1: *Let $\gamma > 0$ be a given constant and $g(t)$ a given positive continuous function. Suppose $t \mapsto u(t)$ satisfies the second order differential inequality*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}u \leq \gamma^2 u + g(t) \tag{231}$$

for all $t \in \mathbb{R}_+$, with $u(0) = u_0 \geq 0$ and $u'(0) = v_0 \geq 0$. Then u is bounded by

$$u(t) \leq u_0 \cosh(\gamma t) + v_0 \frac{1}{\gamma}\sinh(\gamma t) + \int_0^t \cosh[\gamma(t-\tau)]\int_0^\tau g(\tilde{\tau})\mathrm{d}\tilde{\tau}\,\mathrm{d}\tau \tag{232}$$

for all $t \in \mathbb{R}_+$.

Proof of Lemma A.1: Denote r.h.s.(232) $= U(t) = U_{hom}(t) + U_{inh}(t)$, where $U_{inh}(t)$ is the term linear in g. By direct computation one verifies that the function $t \mapsto U(t)$ satisfies (231) with "=" instead of $\leq$, and $U(0) = u_0$ and $U'(0) = v_0$. Since the Cauchy problem for (231) with positive data has a unique positive solution, it follows that $u(t) \leq U(t)$ by the usual subsolution argument. Q.E.D.

A.3 Proof of the conservation laws

We prove the conservation laws for the regularized wave gravity Vlasov equations. The laws for the microscopic regularized field & N-body systems are included as a special case. For general background material on conservation laws, see [SuMu74].

Proof of Proposition 4.6. The conservation of $\mathcal{C}^{(\alpha)}(3)$ holds because (62) is a continuity equation in $\mathbb{R}^6$, and because a Hamiltonian vector field is divergence-free. Q.E.D.

Proof of the conservation laws of Theorems 3.2 and 3.3. As to the conservation of $\mathcal{E}(3)$, for the time derivative of the matter energy (i.e. kinetic plus rest) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint\sqrt{1+|p|^2}f(x,p,t)\,\mathrm{d}x\,\mathrm{d}p = \iint\sqrt{1+|p|^2}(\varrho*\nabla\psi)(x,t)\cdot\partial_p f(x,p,t)\,\mathrm{d}p\,\mathrm{d}x$$

$$= -\iint(\varrho*\nabla\psi)(x,t)\cdot vf(x,p,t)\,\mathrm{d}p\,\mathrm{d}x\,, \qquad (233)$$

where we first pulled the time derivative into the integral, then used (62) to rewrite the integrand, noted that x divergences integrate to zero, then integrated by parts w.r.t. p, and finally used that $\partial_p\sqrt{1+|p|^2} = v$ is the velocity of a particle with unit mass, having momentum p. On the other hand, for the wave field energy, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int\left(|\partial_x\psi|^2+|\varpi|^2\right)(x,t)\mathrm{d}x = \int\left(-\partial_x^2\psi+\partial_t\varpi\right)(x,t)\varpi(x,t)\mathrm{d}x$$

$$= -\int\left(\varrho*\int f(.,p,t)\,\mathrm{d}p\right)(x)\,\varpi(x,t)\,\mathrm{d}x$$

$$= -\iint f(x,p,t)(\varrho*\varpi)(x,t)\,\mathrm{d}x\,\mathrm{d}p\,, \qquad (234)$$

where we pulled the time derivative into the integral, used (58) to rewrite the integrand, and invoked Fubini. Finally, for the regularized coupling energy, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint(\varrho*\psi)(x,t)f(x,p,t)\mathrm{d}x\,\mathrm{d}p = \iint(\varrho*\varpi)(x,t)f(x,p,t)\mathrm{d}x\,\mathrm{d}p$$

$$+\iint(\varrho*\psi)(x,t)\,\partial_t f(x,p,t)\mathrm{d}x\,\mathrm{d}p\,. \qquad (235)$$

The last expression in (234) cancels against the first term on r.h.s.(235). It remains to show that the second term on r.h.s.(235) cancels against the final expression in (233).

We use (62) to rewrite the integrand of the second term on r.h.s.(235), note that p divergences integrate to zero, then invoke Fubini and integrations by parts. Thus

$$\iint (\varrho * \psi)(x,t)\partial_t f(x,p,t)\mathrm{d}x\,\mathrm{d}p = -\int \left(\varrho * \int v \cdot \nabla f(\,.\,,p,t)\,\mathrm{d}p\right)(x)\,\psi(x,t)\,\mathrm{d}x$$

$$= -\iint v \cdot \partial_x f(x,p,t)\,\mathrm{d}p\,(\varrho * \psi)\,(x,t)\,\mathrm{d}x$$

$$= -\int \left(\partial_x \cdot \int v f(x,p,t)\,\mathrm{d}p\right)(\varrho * \psi)(x,t)\,\mathrm{d}x$$

$$= \iint v f(x,p,t)\,\mathrm{d}p \cdot \partial_x(\varrho * \psi)(x,t)\,\mathrm{d}x$$

$$= \iint v f(x,p,t)\,\mathrm{d}p \cdot (\varrho * \nabla\psi)(x,t)\,\mathrm{d}x\,. \qquad (236)$$

Thus conservation of the energy $\mathcal{E}$ is proved.

The conservation of $\mathcal{P}(3)$ is shown similarly. For the matter momentum, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint p f(x,p,t)\,\mathrm{d}p\,\mathrm{d}x = \iint p(\varrho * \nabla\psi)(x,t)\cdot\partial_p f(x,p,t)\,\mathrm{d}p\,\mathrm{d}x$$

$$= -\iint(\varrho * \nabla\psi)(x,t)f(x,p,t)\,\mathrm{d}p\,\mathrm{d}x\,, \qquad (237)$$

the last step through integration by parts, using the identity $(\partial_x u(x) \cdot \partial_p)p = \partial_x u(x)$. On the other hand, for the field momentum we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-\int \partial_x\psi(x,t)\varpi(x,t)\mathrm{d}x\right) = -\int \partial_t\varpi(x,t)\partial_x\psi(x,t)\mathrm{d}x$$

$$= -\int\left(\partial_t\varpi - \partial_x^2\psi\right)(x,t)\partial_x\psi(x,t)\mathrm{d}x$$

$$= \int\left(\varrho * \int f(\,.\,,p,t)\mathrm{d}p\right)(x)\,\partial_x\psi(x,t)\,\mathrm{d}x$$

$$= \iint f(x,p,t)(\varrho * \nabla\psi)(x,t)\,\mathrm{d}p\,\mathrm{d}x\,, \qquad (238)$$

where we used the identity $2\varpi\partial_x\varpi = \partial_x\left(|\varpi|^2\right)$ in the first step, and the identity

$$\partial_x\psi\,\partial_x^2\psi = \partial_x \cdot (\partial_x\psi \otimes \partial_x\psi) - \tfrac{1}{2}\partial_x|\partial_x\psi|^2 \qquad (239)$$

in the second step, and noting the vanishing of "surface integrals at infinity." Adding (237) and (238) we see that total momentum $\mathcal{P}$ is conserved.

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As to the conservation of $\mathcal{J}(3)$, for the orbital matter angular momentum we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint x{\times}p\,f(x,p,t)\,\mathrm{d}p\,\mathrm{d}x = -\iint (x{\times}p)\,v\cdot\partial_x f(x,p,t)\,\mathrm{d}p\,\mathrm{d}x$$

$$+\iint (x{\times}p)\,(\varrho*\nabla\psi)(x,t)\cdot\partial_p f(x,p,t)\,\mathrm{d}p\,\mathrm{d}x$$

$$=\iint (\varrho*\nabla\psi)(x,t){\times}x f(x,p,t)\,\mathrm{d}p\,\mathrm{d}x$$

$$=\int \partial_x\psi(x,t){\times}x\,\left(\varrho*\int f(\,.\,,p,t)\mathrm{d}p\right)(x)\,\mathrm{d}x$$

$$-\int \partial_x\psi(x,t){\times}\left(\varrho\mathrm{Id}*\int f(\,.\,,p,t)\,\mathrm{d}p\right)(x)\,\mathrm{d}x\,. \quad (240)$$

In the first step we used the continuity equation (62), in the second step integrations by parts and the identities $(v\cdot\partial_x)(x{\times}p)=v{\times}p=0$ and $(\partial_x u(x)\cdot\partial_p)(x{\times}p)=x{\times}\partial_x u(x)$; the last step is Fubini and a trivial rewriting. The last integral in (240) gives

$$-\int \partial_x\psi(x,t){\times}\left(\varrho\mathrm{Id}*\int f(\,.\,,p,t)\mathrm{d}p\right)(x)\mathrm{d}x=\iint (\varrho\mathrm{Id}*{\times}\nabla\psi(\,.\,,t))(x)f(x,p,t)\mathrm{d}p\mathrm{d}x \quad (241)$$

where again we used Fubini. Finally, by some standard identities of vector analysis and the radial symmetry of ϱ, the (negative of the) field torque evaluates to

$$(\varrho\mathrm{Id}*{\times}\nabla\psi(\,.\,,t))\,(x)=\int \partial_y{\times}\Big((y-x)\varrho(y-x)\psi(y,t)\Big)\mathrm{d}y=0\,. \quad (242)$$

The last integral vanishes by one of Green's theorems and the compact support of ϱ.

Lastly, for the field angular momentum we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (-x{\times}\partial_x\psi(x,t))\varpi(x,t)\mathrm{d}x=-\int (x{\times}\partial_x\psi(x,t))\partial_t\,\varpi(x,t)\mathrm{d}x$$

$$=\int (x{\times}\partial_x\psi(x,t))\,(\partial_x^2\psi-\partial_t\,\varpi)(x,t)\mathrm{d}x$$

$$=-\int \partial_x\psi(x,t){\times}x\left(\varrho*\int f(\,.\,,p,t)\mathrm{d}p\right)(x)\mathrm{d}x \quad (243)$$

where we used the identity $2\varpi x{\times}\partial_x\varpi=\partial_x{\times}\left(x|\varpi|^2\right)$ in the first step, and the identity

$$(x{\times}\partial_x\psi)\,\partial_x^2\psi=\partial_x\cdot((x{\times}\partial_x\psi)\otimes\partial_x\psi)-\partial_x{\times}\left(\tfrac{1}{2}\,x\,|\partial_x\psi|^2\right) \quad (244)$$

in the second, and (58) in the third, noting the vanishing of "surface integrals."

Adding (240) and (243), noting (241), (244), proves conservation of $\mathcal{J}$. Q.E.D.

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